Math 432: Set Theory and Topology HOMEWORK 13 **Optional**

1. Prove that continuous functions map compact sets to compact sets; more precisely, letting X, Y be metric spaces and $f : X \to Y$ continuous, prove that if X is compact then $f(X)$ is compact.

2. Prove the familiar(?) statement from calculus that continuous functions on closed intervals are bounded and attain their maximum and minimum. More generally, let X be a compact metric space and let $f : X \to \mathbb{R}$ be continuous, and prove that $f(X)$ is bounded and attains its maximum and minimum.

3. In Homework 12, you proved (1) $\Leftrightarrow$ (2) of the following.

Theorem. *For a metric space X, the following are equivalent:*

(1) *X is separable, i.e. has a countable dense set.*

(2) *X admits a countable open base.*

(3) *Every open cover of X has a countable subcover.*

Now prove (2) $\Leftrightarrow$ (3) as follows.

(2)$\Rightarrow$(3): Let $\mathcal{C}$ be an open cover and let $\mathcal{B}$ be a countable base for X. Put

$$\mathcal{B}' := \{U \in \mathcal{B} : \exists V \in \mathcal{C} \text{ with } V \supseteq U\}$$

and for each $U \in \mathcal{B}'$, choose (by Axiom of Choice) a set $V_U \in \mathcal{C}$ with $V_U \supseteq U$. Show that $\mathcal{C}' := \{V_U : U \in \mathcal{B}'\}$ is a cover of X.

(3)$\Rightarrow$(2): For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the collection of all open balls of radius $\frac{1}{n+1}$. Clearly $\mathcal{C}_n$ is a cover, so there is a countable subcover $\mathcal{C}'_n$. Show that $\mathcal{B} := \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ is a countable base for X.