MATH 570: MATHEMATICAL LOGIC

POTENTIAL FINAL EXAM PROBLEMS

1. Let $(L,<)$ be a linearly ordered set and let $(\boldsymbol{A}_\ell)_{\ell \in L}$ be an increasing elementary chain of σ-structures, i.e. $\boldsymbol{A}_\ell \preceq \boldsymbol{A}_k$ for $\ell \leq k$. Show that $A = \bigcup_{\ell \in L} A_\ell$ is a universe of a σ-structure $\boldsymbol{A}$ and that $\boldsymbol{A}_\ell \preceq \boldsymbol{A}$ for each $\ell \in L$.

2. Let $\boldsymbol{M}$ be a σ-structure.

Definition. $\boldsymbol{M}$ is said to be *weakly homogeneous* if for every pair $\boldsymbol{A} \subseteq \boldsymbol{B}$ of finitely generated substructures of $\boldsymbol{M}$, every embedding (not necessarily elementary) $h : \boldsymbol{A} \hookrightarrow \boldsymbol{M}$ extends to an embedding $\bar{h} : \boldsymbol{B} \hookrightarrow \boldsymbol{M}$.

Definition. For a subset $A \subseteq \boldsymbol{M}$ and a σ-structure $\boldsymbol{K}$, a function $h : A \to \boldsymbol{K}$ is called a *partial elementary map* $\boldsymbol{M} \rightharpoonup_e \boldsymbol{K}$ if for every σ-formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$,

$$\boldsymbol{M} \models \varphi(\vec{a}) \text{ if and only if } \boldsymbol{K} \models \varphi(h(\vec{a})).$$

Definition. For an infinite cardinal κ, $\boldsymbol{M}$ is said to be *κ-homogenenous* if for every $A \subseteq M$ with $|A| < \kappa$ and $a \in M$, every partial elementary map $h : A \to M$ extends to a partial elementary map $\bar{h} : A \cup \{a\} \to M$.

Prove that if $\boldsymbol{M}$ is weakly homogeneous, then for every finitely generated substructure $\boldsymbol{A} \subseteq \boldsymbol{M}$, any embedding $\boldsymbol{A} \hookrightarrow \boldsymbol{M}$ is a partial elementary map $\boldsymbol{M} \rightharpoonup \boldsymbol{M}$. Deduce that weak homogeneity implies $\aleph_0$-homogeneity.

3. Give an example of a structure $\boldsymbol{A}$ (in some signature) and a definable binary relation Q in it such that the unary relation

$$P(y) :\Leftrightarrow \exists^\infty x \; Q(x,y)$$

is not definable in $\boldsymbol{A}$, where "$\exists^\infty x$" means "for infinitely many x". Provide a proof of every statement you use that claims nondefinability of a set.

4. Let $\sigma := (E)$ be the signature of graphs, i.e. E is a binary relation symbol.

(a) Write down an explicit axiomatization T for the class of undirected graphs with no loops, whose connected components are *bi-infinite chains*, i.e. acyclic graphs with the degree of each vertex being 2.

(b) Show that T is complete.

(c) Conclude yet again (for the last time, I promise) that the binary relation R of being in the same connected component is not 0-definable in any disconnected model of T.

Hint: Let $\varphi_R(x,y)$ be a formula defining R in a disconnected model $\boldsymbol{M}$, so $\boldsymbol{M} \models \exists x \exists y \neg\varphi_R(x,y)$, hence $\boldsymbol{Z} \models \exists x \exists y \neg\varphi_R(x,y)$, where $\boldsymbol{Z}$ is the connected model. Elements $a,b \in Z$ that witness the latter sentence are within finite distance from each other...

(d) Show that for any $\boldsymbol{M} \models T$ and $a,b \in M$, there is an automorphism g of $\boldsymbol{M}$ with $g(a) = b$.

(e) For $\boldsymbol{M} \models T$, exactly which subsets of M are 0-definable in $\boldsymbol{M}$?

(f) Finally, prove that T is model-complete, but does not admit q.e. Recall that that $\mathbb{FOL}(\sigma)$ includes the 0-ary relation symbols for truth and falsehood, so σ not having a constant symbol is not the reason why T doesn't admit q.e.

5. Consider the following function:

$$f(n) = \begin{cases} 1 & \text{if the decimal expression of } \pi \text{ contains } n \text{ consecutive zeroes} \\ 0 & \text{otherwise} \end{cases}.$$

Is it Σ_1^0? Recursive? Primitive recursive? You may assume that the function that outputs the n^{th} decimal digit of π is primitive recursive.

6. Give an example of a Σ_1^0 unary relation that is not recursive. Justify your answer.

7. For each of the following, either prove that it holds for all σ_{arthm}-sentences θ or provide an example of a θ for which it fails. Justify your answers.

(a) $\mathrm{PA} \models \theta \to \mathbf{Provable}_{\mathrm{PA}}([\theta])$.

(b) $\mathrm{PA} \models \mathbf{Provable}_{\mathrm{PA}}([\theta]) \to \theta$.

(c) If $\boldsymbol{N} \models \neg\mathbf{Provable}_{\mathrm{PA}}([\theta])$ then $\mathrm{PA} \models \neg\mathbf{Provable}_{\mathrm{PA}}([\theta])$.

(d) If $\mathrm{PA} \models \neg\mathbf{Provable}_{\mathrm{PA}}([\theta])$ then $\boldsymbol{N} \models \neg\mathbf{Provable}_{\mathrm{PA}}([\theta])$.

8. Let $A, B \subseteq \mathbb{N}^k$. Prove:

(a) Reduction property for Σ_1^0: If A, B are Σ_1^0, then there are disjoint Σ_1^0 sets $A^*, B^* \subseteq \mathbb{N}^k$ such that $A^* \subseteq A$, $B^* \subseteq B$ and $A^* \cup B^* = A \cup B$.

HINT: For $\vec{a} \in \mathbb{N}^k$, decide whether to put it in A^* or B^* based on which of A and B claims it first (i.e. has the smaller witness).

(b) Separation property for Π_1^0: If A, B are disjoint Π_1^0 sets, then there is a Δ_1^0 (and hence recursive) set $S \subseteq \mathbb{N}^k$ such that $S \supseteq A$ and $S^c \supseteq B$.

9. Let $n, k \in \mathbb{N}$.

(a) Construct universal sets for $\Sigma_n^0(\mathbb{N}^k)$ and $\Pi_n^0(\mathbb{N}^k)$.

HINT: This is in the notes for $n = 1$ and the rest is by induction.

(b) Prove that $\Delta_n^0(\mathbb{N}^k)$ does not admit a universal set.

(c) Deduce that $\Delta_n^0(\mathbb{N}^k) \subsetneq \Sigma_n^0(\mathbb{N}^k) \subsetneq \Delta_{n+1}^0(\mathbb{N}^k)$ and $\Delta_n^0(\mathbb{N}^k) \subsetneq \Pi_n^0(\mathbb{N}^k) \subsetneq \Delta_{n+1}^0(\mathbb{N}^k)$. Make sure to also show the inclusions, not just their strictness.

(d) Show that $\bigcup_n \Sigma_n^0 = \bigcup_n \Delta_n^0 = \bigcup_n \Pi_n^0$ is precisely the class of all arithmetical sets.

(e) Conclude Tarski's theorem that $\ulcorner\mathrm{Th}(\boldsymbol{N})\urcorner$ is not arithmetical, where $\boldsymbol{N} := (\mathbb{N}, 0, S, +, \cdot)$.

10. Prove that for a σ-theory T, the following are equivalent:

(1) T is model-complete.

(2) For every model $\boldsymbol{A} \models T$, $T \cup \mathrm{Diag}(\boldsymbol{A})$ is a complete σ_A-theory.

(3.a) Every σ-formula $\varphi(\vec{x})$ is equivalent in T to a universal formula.

(3.b) Every σ-formula $\varphi(\vec{x})$ is equivalent in T to an existential formula.

Hint: For (2)⇒(3.a), mimic the proof of "diagram-complete $\implies$ q.e." More precisely, consider the set

$$\Gamma(\vec{x}) := \{\psi : \psi(\vec{x}) \text{ is a universal } \sigma\text{-formula and } T \models \varphi \to \psi\}$$

and show that $T \cup \Gamma(\vec{x}) \models \varphi$ (the Generalization axiom is involved).

11. Let $\mathcal{C}$ be a class (a set) of σ-structures. Define the *theory* $\mathrm{Th}(\mathcal{C})$ and the *asymptotic theory* $\mathrm{Th}_a(\mathcal{C})$ of $\mathcal{C}$ as follows: for every σ-sentence φ,

$$\varphi \in \mathrm{Th}(\mathcal{C}) :\Leftrightarrow \forall \boldsymbol{M} \in \mathcal{C} \;\; \boldsymbol{M} \models \varphi,$$
$$\varphi \in \mathrm{Th}_a(\mathcal{C}) :\Leftrightarrow \forall^\infty \boldsymbol{M} \in \mathcal{C} \;\; \boldsymbol{M} \models \varphi,$$

where $\forall^\infty$ means "for all but finitely many".

(a) Let $\mathcal{C}$ be an infinite class of finite structures that contains only finitely many structures of cardinality n, for each $n \in \mathbb{N}$. Prove that the models of $\mathrm{Th}_a(\mathcal{C})$ are exactly the infinite models of $\mathrm{Th}(\mathcal{C})$.

Hint: To prove that any infinite model $\boldsymbol{M} \models \mathrm{Th}(\mathcal{C})$ is a model of $\mathrm{Th}_a(\mathcal{C})$, note that if $\varphi \in \mathrm{Th}_a(\mathcal{C})$, then $\psi \to \varphi \in \mathrm{Th}_a(\mathcal{C})$ for any ψ. Find a suitable ψ that is true in $\boldsymbol{M}$ and $\psi \to \varphi \in \mathrm{Th}(\mathcal{C})$.

(b) Let $\sigma_{\mathrm{gr}} := (E)$ be the signature of graphs and for each $n \geq 2$, let $\boldsymbol{G}_n$ be the (undirected) graph that is a chain (path) of n vertices, i.e. $\boldsymbol{G}_n := (V_n, E^{\boldsymbol{G}_n})$, where $V_n := \{1, 2, \dots, n\}$ and $u E^{\boldsymbol{G}_n} v :\Leftrightarrow |u - v| = 1$, for $u, v \in V_n$. Let $\mathcal{C} := \{\boldsymbol{G}_n : n \geq 2\}$. Let

$$T := \{\chi, \theta\} \cup \{\varphi_n, \psi_n : n \geq 2\},$$

where

- χ says that E is irreflexive and symmetric.
- θ says that there are exactly two vertices with degree 1 (call them *leaves*) and all other vertices have degree 2.
- φ_n says that there are two leaves whose distance is at least n.
- ψ_n says that there are no cycles of length n.

Prove that $T \subseteq \mathrm{Th}_a(\mathcal{C})$.

(c) Prove that T is complete.

(d) Conclude that T is an axiomatization of $\mathrm{Th}_a(\mathcal{C})$.

(e) Describe all models of $\mathrm{Th}_a(\mathcal{C})$.