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Math 570: Mathematical Logic HOMEWORK 9 **Due: Nov 16–17**

1. (a) (Optional) Show that the Ackermann function grows faster than any primitive recursive function; more precisely, prove that for any primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists $n_f \in \mathbb{N}$ such that $f(\vec{x}) \leq A(n_f, \|\vec{x}\|_1)$ for all $\vec{x} \in \mathbb{N}^k$, where $\|\vec{x}\|_1 := x_1 + \ldots + x_n$.

HINT: Prove by induction on the construction of f from the basic functions. You may use all of the properties of the Ackermann function stated in the previous homework even if you did not prove them.

REMARK: This problem has less priority than the others, I suggest doing this last.

(b) Conclude that the Ackermann function is not primitive recursive.

2. Follow the outline below to prove-sketch that the class $\mathcal{R}_0(\mathbb{N})$ of all primitive recursive functions from $\mathbb{N}$ to $\mathbb{N}$ admits a recursive $\mathbb{N}$-parameterization $\Lambda : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Recall that $\langle \cdot \rangle_k$ denotes the k-tuple encoding function and observe that is satisfies the condition

$$\langle \vec{a} \rangle \geq \max\{k, a_0, \ldots, a_{k-1}\}$$

for each $\vec{a} \in \mathbb{N}^k$. Below, we omit the subscript k. Also, we say a function $f : \mathbb{N}^k \to \mathbb{N}$ *is encoded* by $f' : \mathbb{N} \to \mathbb{N}$ if for each $\vec{a} \in \mathbb{N}^k$, $f'(\langle \vec{a} \rangle_k) = f(\vec{a})$.

Firstly, it is enough to define a recursive function $\Upsilon : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ whose fibers encode all of the primitive recursive functions, i.e. for each primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$ there is $n \in \mathbb{N}$ such that Υ_n encodes f. Indeed, one then obtains a recursive $\mathbb{N}$-parameterization of $\mathcal{R}_0(\mathbb{N})$ by taking $\Lambda(n, a) := \Upsilon(n, \langle a \rangle)$.

For obtaining such a function Υ, it is enough to have a recursive function $\Upsilon : \mathbb{N}^2 \to \mathbb{N}$ satisfying the following for every $n \in \mathbb{N}$:

- if $n = \langle 0, d, m \rangle$ then Υ_n encodes the successor function $S : \mathbb{N} \to \mathbb{N}$.
- if $n = \langle 1, d, m \rangle$ then Υ_n encodes projection function $P_m^{(d)} : \mathbb{N}^d \to \mathbb{N}$.
- if $n = \langle 2, d, m \rangle$ then Υ_n encodes constant function $C_m^{(d)} : \mathbb{N}^d \to \mathbb{N}$.
- if $n = \langle 3, d, m \rangle$, where
  - $m = \langle n_0, \ldots, n_k \rangle$ for some $k \geq 1$,
  - $n_0 = \langle c_0, k, m_0 \rangle$,
  - $n_i = \langle c_i, d, m_i \rangle$ for each $i = 1, \ldots, k$,

  then, letting $g : \mathbb{N}^k \to \mathbb{N}$ be the function encoded by Υ_{n_0} and $h_i : \mathbb{N}^d \to \mathbb{N}$ the functions encoded by Υ_{n_i}, Υ_n encodes the function $g(h_1, \ldots, h_k)$ obtained by *composition* from g and $h_1, \ldots, h_k$.
- if $n = \langle 4, d, m \rangle$, where
  - $d \geq 1$

- $m = \langle n_0, n_1 \rangle$
- $n_0 = \langle j_0, d-1, m_0 \rangle$,
- $n_1 = \langle j_1, d+1, m_1 \rangle$,

  then letting $g : \mathbb{N}^{d-1} \to \mathbb{N}$ be the function encoded by Υ_{n_0} and $h : \mathbb{N}^{d+1} \to \mathbb{N}$ the function encoded by Υ_{n_1}, Υ_n encodes the function obtained by *primitive recursion* from g and h.

To define the value $\Upsilon(n,l)$, one needs to know $\Upsilon(n',l')$ for only finitely many (n',l') with either $n' < n$ or $l' < l$. Use this and Dedekind's analysis of recursion to define a recursive Υ satisfying all the conditions above. This is done as described in the hint for Question 4 of Homework 8 on the recursivity of the Ackermann function.

3. Prove that all recursive functions and relations are arithmetical, i.e. definable in $\boldsymbol{N} := (\mathbb{N}, 0, S, +, \cdot)$.

4. (a) Show that for any theory $T \subseteq \mathrm{Th}(\boldsymbol{N})$, the functions and relations representable in T are arithmetical.

(b) Show that the converse is true for $T := \mathrm{Th}(\boldsymbol{N})$.

(c) Do you think the converse is true for PA even just for relations? More precisely, is every relation definable in $\boldsymbol{N}$ representable in PA? What might be a potential issue?

5. Let T be a σ_{arthm}-theory satisfying $T \models \Delta(n) \neq \Delta(m)$ for all distinct $n, m \in \mathbb{N}$.

(a) Prove in detail that if a function f is representable in T by a formula φ then the same formula represents the graph of f. (This was proven in class.)

(b) Do you think the converse is true for $T := \mathrm{Th}(\boldsymbol{N})$? What might be a potential issue?