Math 570: Mathematical Logic HOMEWORK 8 **Due: Nov 9–10**

1. Prove Lemma 5.8(f) as well as Lemma 5.11(a,b,c).

Definition. The class of *primitive recursive* functions is the smallest class containing the successor function $S : \mathbb{N} \to \mathbb{N}$, the constant functions $C_m^k : \mathbb{N}^k \to \mathbb{N}$, $k, m \in \mathbb{N}$ and the projection functions $P_i^k(x_1, \ldots, x_k) = x_i$, $i \leq k \in \mathbb{N}$, and is closed under composition (R2) and primitive recursion (R4). A relation $R \subseteq \mathbb{N}^k$ is called *primitive recursive* if its characteristic function $\mathbf{1}_R : \mathbb{N}^k \to \mathbb{N}$ is primitive recursive.

Definition. A function $f : \mathbb{N}^k \to \mathbb{N}$ is called *primitive recursive* if it is either one of the functions in (R1) or it is obtained from them via finitely-many applications of the operations of composition (R2) and primitive recursion (R4).

Below, let $\beta(w, i)$ denote Gödel's β-function for encoding arbitrarily long tuples of natural numbers; see Lemma 5.14. You may assume below that Gödel's β-function and (consequently) all functions in (5.15.i–v) are primitive recursive.

2. For $f : \mathbb{N}^{k+1} \to \mathbb{N}$, define

$$\bar{f}(\vec{a}, n) := \langle f(\vec{a}, 0), f(\vec{a}, 1), \ldots, f(\vec{a}, n-1) \rangle_n .$$

Given $h : \mathbb{N}^{k+1} \to \mathbb{N}$, let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be defined by the identity

$$f(\vec{a}, n) := h(\vec{a}, \bar{f}(\vec{a}, n)).$$

Show that if h is primitive recursive, then f too is primitive recursive.

3. (Optional) Let $g : \mathbb{N} \to \mathbb{N}$, $h : \mathbb{N}^3 \to \mathbb{N}$, $\tau : \mathbb{N}^2 \to \mathbb{N}$. We say that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by *nested recursion* from g, h, τ if for each $x, y \in \mathbb{N}$,

$$\begin{cases} f(0, y) & := & g(y) \\ f(x+1, y) & := & h(x, y, f(x, \tau(x, y))). \end{cases}$$

Show that if g, h, τ are primitive recursive, then so is f.

HINT: This is not an easy problem: one has to define an auxiliary function first, prove it is primitive recursive, and then express f primitive-recursively in terms of the auxiliary function.

Definition. The *Ackermann* function $A : \mathbb{N}^2 \to \mathbb{N}$ is inductively defined as follows:

$$\begin{cases} A(0, x) & := & x + 1 \\ A(n+1, 0) & := & A(n, 1) \\ A(n+1, x+1) & := & A(n, A(n+1, x)). \end{cases}$$

4. Prove that the Ackermann function is recursive.

HINT: To prove that primitive recursion operation is a recursive operation, we used the Dedekind analysis of primitive recursion, namely, we *searched* for a tuple encoding all of the previous values of the function. In this tuple, the i^{th} term was equal to the value of the function at i; thus, we could equivalently search for a tuple of pairs (i, w), where w would be the value of the function at i. For the Ackermann function, a similar analysis can be done, where we again *search* for a tuple of triples (i, j, w), where we put appropriate conditions to ensure that $w = A(i, j)$. In other words, this tuple we are looking for is a **partial** matrix of values of A.

5. (Optional[1]) Show that the graph of the Ackermann function is primitive recursive. This implies, once again, that the Ackermann function itself is recursive.

Hint: Use a similar argument to the proof that the Ackermann function is recursive, but bound your search using the value of the function.

6. Below are some properties of the Ackermann function that will be used in the next homework in showing that the Ackermann function is not primitive recursive. Choose any two of these properties and prove them.

(a) $A(n, x+y) \geq A(n, x) + y$.

(b) For $n \geq 1$, $A(n+1, y) > A(n, y) + y$.

(c) $A(n+1, y) \geq A(n, y+1)$.

(d) $2A(n, y) < A(n+2, y)$.

(e) If $x < y$ then $A(n, x+y) \leq A(n+2, y)$.

Hint: Induct on the sum of both coordinates (n, m) or on the lexicographical ordering on $\mathbb{N}^2$.

[1] Yet recommended.