Math 570: Mathematical Logic HOMEWORK 6 **Due:** Oct 26–27

1. The following is a well known theorem of additive combinatorics:

Theorem (van der Waerden)**.** *Suppose $\mathbb{N}$ is finitely colored. Then one of the color classes contains arbitrarily long arithmetic progressions.*

Use this theorem and the Compactness theorem to derive the following finitary version:

Theorem. *Given any positive integers m and k, there exists $N \in \mathbb{N}$ such that whenever $\{0, 1, ..., N-1\}$ is colored with m colors, one of the color classes contains an arithmetic progression of length k.*

2. Let $\boldsymbol{G}_2 := (V, E)$ be the graph consisting of 2 bi-infinite paths (i.e. $\mathbb{Z}$-lines), denote these paths by A and B. Prove that the connectedness relation $R \subseteq V^2$ (i.e. the relation between two vertices of being connected by a path) is not definable in $\boldsymbol{G}_2$ as follows: suppose towards a contradiction that there is a formula $\varphi(x, y, \vec{z})$ and a vector $\vec{p} \subseteq V^{|\vec{p}|}$ such that $R = \left\{(a, b) \in V^2 : \boldsymbol{G}_2 \models \varphi(a, b, \vec{p})\right\}$. Build an elementary extension of $\boldsymbol{G}_2$ containing two other bi-infinite paths C, D such that $\varphi(x, y, \vec{p})$ holds between elements of C and A but doesn't hold between elements of D and A. Obtain a contradiction by swapping C and D.

3. For $k \geq 1$, let $\boldsymbol{G}_k$ be the graph consisting of k-many bi-infinite paths.

(a) Prove that there is a countable elementary extension $\overline{\boldsymbol{G}}$ of $\boldsymbol{G}_k$ that has infinitely-many connected components.

(b) Prove that any such $\overline{\boldsymbol{G}}$ is isomorphic to the graph $\boldsymbol{G}_{\aleph_0}$ consisting of $\aleph_0$-many bi-infinite paths.

(c) Deduce that $\boldsymbol{G}_\ell \preceq \boldsymbol{G}_k$, for any $1 \leq \ell \leq k$.

(d) Deduce further that the connectedness relation between two vertices is not 0-definable in $\boldsymbol{G}_k$, for any $k \in \{2, 3, \ldots\} \cup \{\aleph_0\}$.

4. Prove directly, without referring to anything proven in lecture, that the class of disconnected graphs is not axiomatizable.

5. Let DLO denote the theory of dense linear orderings without endpoints.

(a) Show that DLO is $\aleph_0$-categorical, and hence $(\mathbb{Q}, <)$ is the only (up to isomorphism) countable dense linear ordering without endpoints.

HINT: Enumerate the two models and recursively construct a sequence of finite partial isomorphisms by going back and forth between the models.

(b) Let $\sigma_n := (<, c_1, ..., c_n)$, where c_i are constant symbols. Show that the theory

$$\mathrm{DLO}_n := \mathrm{DLO} \cup \{c_i < c_{i+1} : i < n\}$$

is $\aleph_0$-categorical. Conclude that DLO_n is complete.

(c) Let $\sigma_\infty = (<, \{c_i : i \in \mathbb{N}\})$, where c_i are constant symbols. Show that the theory

$$\mathrm{DLO}_\infty = \mathrm{DLO} \cup \{c_i < c_{i+1} : i \in \mathbb{N}\}$$

has exactly three countable nonisomorphic models and hence is not $\aleph_0$-categorical.

(d) Yet, prove that DLO_∞ is complete.