Math 570: Mathematical Logic HOMEWORK 5 **Due: Oct 19–20**

1. Let T be a σ-theory.

   (a) We say that T *induces* the subset $\langle T\rangle := \bigcap_{\phi\in T}\langle\phi\rangle$ of the topological space $\mathcal{T}_\sigma$ of all maximally σ-complete satisfiable theories. What does the statement that T is finitely axiomatizable correspond to in terms of $\langle T\rangle$ topologically? In other words, fill in the blank in the following proposition and prove it.

   Proposition. *T is finitely axiomatizable if and only if $\langle T\rangle$ is a _____ subset of $\mathcal{T}_\sigma$.*

   (b) Show that for every finitely axiomatizable theory T contains a finite axiomatization of itself, i.e. there is a finite subset $T_0 \subseteq T$ with $\boldsymbol{M}_\sigma(T_0) = \boldsymbol{M}_\sigma(T)$.

2. A graph $\boldsymbol{G} := (V, E)$ is called *bipartite* if there is a partition $V = V_1 \uplus V_2$ into disjoint nonempty parts V_1, V_2 such that there is no edge between the vertices in the same part, i.e. $E \cap V_i^2 = \emptyset$ for each $i \in \{1, 2\}$. Note that this definition is not first-order as it involves quantification over subsets of the underlying set.

   (a) Nevertheless, show that the class of bipartite graphs is axiomatizable.

   HINT: Find/google an equivalent condition that is first-order.

   (b) However, show that the class of bipartite graphs is not finitely axiomatizable.

3. Prove that in an undirected locally finite[1] graph $\boldsymbol{G} := (V, E)$, if every finite subgraph is contained in a finite subgraph that admits a perfect matching[2], then the entire graph $\boldsymbol{G}$ admits a perfect matching.

4. Show that the following classes of structures are not axiomatizable:

   (a) cyclic groups,

   HINT: Think about cardinalities of models of a hypothetical axiomatization.

   (b) torsion groups,

   (c) connected graphs.

   HINT: Add two new constant symbols c_1, c_2 to the signature, and, given a hypothetical axiomatization, slowly add new sentences to it regarding the graph-distance between c_1 and c_2, so that the resulting theory is still finitely satisfiable, but its models are necessarily disconnected graphs.

5. Let $\boldsymbol{M} \models \mathrm{PA}$.

   (a) Let $\boldsymbol{N} := (\mathbb{N}, 0^{\boldsymbol{N}}, S^{\boldsymbol{N}}, +^{\boldsymbol{N}}, \cdot^{\boldsymbol{N}})$ denote the standard model of PA, i.e. the usual natural numbers. Show that there is a unique homomorphism $f : \boldsymbol{N} \to \boldsymbol{M}$ and this f is one-to-one, and hence (why?) is a σ_{arthm}-embedding.

[1] A graph is called *locally finite* if every vertex has only finitely many neighbors.

[2] In an undirected graph $\boldsymbol{G} := (V, E)$, a *perfect matching* is a subset $P \subseteq E$ of (undirected) edges such that each vertex $v \in V$ is incident to exactly one edge in P.

(b) In the notation of (a), define the standard part of $\boldsymbol{M}$ by

$$\mathbb{N}^{\boldsymbol{M}} = f[\mathbb{N}].$$

Show that if $\boldsymbol{M}$ is nonstandard then $\mathbb{N}^{\boldsymbol{M}}$ is not definable in $\boldsymbol{M}$.

Hint: Induction schema

(c) (Overspill) Assume $\boldsymbol{M}$ is nonstandard, and let $\phi(x, \vec{y})$ be a σ_{arthm}-formula, where $\vec{y}$ is a k-vector and $\vec{a} \in M^k$. Show that if $\boldsymbol{M} \models \phi(n, \vec{a})$ for infinitely many $n \in \mathbb{N}^{\boldsymbol{M}}$, then there is $w \in M \setminus \mathbb{N}^{\boldsymbol{M}}$ such that $\boldsymbol{M} \models \phi(w, \vec{a})$. In other words, if a statement is true about infinitely many natural numbers, then it is true about an infinite[3] number.

6. Give an example of structures $\boldsymbol{A} \preceq \boldsymbol{B}$ that demonstrate the failure of the converse of

Problem 2 of HW3. For σ-structures $\boldsymbol{A} \subseteq \boldsymbol{B}$, if for any finite $P \subseteq A$ and $b \in B$, there exists an automorphism f of $\boldsymbol{B}$ that fixes P pointwise and $f(b) \in A$, then $\boldsymbol{A} \preceq \boldsymbol{B}$.

7. This problem illustrates the (crazy) structure of the nonstandard models of PA. Let $\boldsymbol{M}$ be a nonstandard model of PA and let $\mathbb{N}^{\boldsymbol{M}}$ be its standard part. By replacing it with $\mathbb{N}$, we can assume without loss of generality that $\mathbb{N}^{\boldsymbol{M}} = \mathbb{N}$.

(a) For $a, b \in M$, let $|a - b|$ denote the unique $d \in M$ such that $a + d = b$ or $b + d = a$ (why does such exist?). For all $a, b \in M$, define

$$a \sim b :\Leftrightarrow |a - b| \in \mathbb{N}.$$

Show that $\sim$ is an equivalence relation on M and that it is NOT definable in $\boldsymbol{M}$.

(b) Put $Q = M/\sim$, so $Q = \{[a] : a \in M\}$, where $[a]$ denotes the equivalence class of a. Define the relation $<_Q$ on Q as follows: for all $[a], [b] \in Q$,

$$[a] <_Q [b] :\Leftrightarrow \exists c \in M \setminus \mathbb{N} \text{ such that } a + c = b$$

Show that $<_Q$ is well-defined (does not depend on the representatives a, b) and is a linear ordering on Q.

(c) Show that the ordering $(Q, <_Q)$ has a least element but no greatest element; moreover, show that it is a *dense* (in itself), i.e. for all $u, v \in Q$,

$$u <_Q v \Rightarrow \exists w (u <_Q w <_Q v).$$

[3] An element $a \in M$ is said to be *infinite* if it is greater than every standard natural number.