Math 570: Mathematical Logic HOMEWORK 4 **Due: Oct 5–6**

1. (Weak Lefschetz Principle) Let ϕ be a τ_{ring}-sentence. Show that if $\text{ACF}_0 \vdash \phi$, then for large enough primes p, $\text{ACF}_p \vdash \phi$.

2. Fill in the details of the proofs of (d), (e), (f) of Proposition 2.12.

3. Fill in the details of the proof of the Constant Substitution lemma.

4. Let $\mathcal{T}_\sigma^{\text{Con}}$ be the set of maximally complete consistent theories. Denote by $\mathcal{D}_\sigma$ the collection of the subsets of $\mathcal{T}_\sigma^{\text{Con}}$ of the form $\langle\phi\rangle := \left\{T \in \mathcal{T}_\sigma^{\text{Con}} : T \ni \phi\right\}$, where ϕ is a σ-sentence. Equip $\mathcal{T}_\sigma^{\text{Con}}$ with the topology generated by $\mathcal{D}_\sigma$ and prove that it is zero-dimensional compact Hausdorff.

REMARK: Recall the space $\mathcal{T}_\sigma^{\text{Sat}}$ of maximally complete satisfiable theories (defined in Homework 3). By definition, $\mathcal{T}_\sigma^{\text{Sat}}$ is a subspace of $\mathcal{T}_\sigma^{\text{Con}}$, i.e. the topology on $\mathcal{T}_\sigma^{\text{Sat}}$ defined in Homework 3 (trivially) coincides with the relative topology of $\mathcal{T}_\sigma^{\text{Con}}$. The Completeness theorem is equivalent to $\mathcal{T}_\sigma^{\text{Sat}} = \mathcal{T}_\sigma^{\text{Con}}$.

5. Pick any two of the following statements and prove them. You may assume the preceding statements in your proofs.

(a) (0 is also a left-identity) $\text{PA} \vdash \forall x(0 + x \doteq x)$.

(b) (Associativity of +) $\text{PA} \vdash \forall x \forall y \forall z\big((x+y)+z \doteq x+(y+z)\big)$.

(c) (Commutativity of +) $\text{PA} \vdash \forall x \forall y(x+y \doteq y+x)$.

6. Consider the following (finite) theory, the so called Robinson's system Q:

(Q1) $\forall x(\neg S(x) \doteq 0)$,

(Q2) $\forall x \forall y(S(x) \doteq S(y) \to x \doteq y)$,

(Q3) $\forall x(x+0 \doteq x)$,

(Q4) $\forall x \forall y(S(x+y) \doteq x+S(y))$,

(Q5) $\forall x(x \cdot 0 \doteq 0)$,

(Q6) $\forall x \forall y(x \cdot S(y) \doteq x \cdot y + x)$,

(Q7) $\forall x\Big(x \not\doteq 0 \to \exists y\big[x \doteq S(y)\big]\Big)$.

Show:

(a) $\text{Q} \nvDash \forall x \forall y \forall z\big[(x+y)+z \doteq x+(y+z)\big]$.

(b) $\text{Q} \nvDash \forall x(0 + x \doteq x)$.

(c) $\text{Q} \nvDash \forall x \forall y(x+y \doteq y+x)$.