Math 570: Mathematical Logic HOMEWORK 3 **Due: Sept 28–29**

1. (Chasing definitions) Working in signature σ, prove:

(a) A theory is semantically complete if and only if any two models of it are elementarily equivalent.

(b) Every satisfiable theory admits a satisfiable maximal completion.

(c) Every satisfiable semantically complete theory admits a unique satisfiable maximal completion.

2. The following is a very useful sufficient condition for being an elementary substructure. Let $\boldsymbol{A} \subseteq \boldsymbol{B}$ and assume that for any finite $P \subseteq A$ and $b \in B$, there exists an automorphism f of $\boldsymbol{B}$ that fixes P pointwise (i.e. $f(p) = p$ for all $p \in P$) and $f(b) \in A$. Show that $\boldsymbol{A} \preceq \boldsymbol{B}$.

3. Show that $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$. Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$, but $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$.

HINT: Use Exercise 2.

4. Let $\boldsymbol{B} = (B, E)$ be a countable graph, each of whose vertices have degree at most 1. Suppose further that $\boldsymbol{B}$ has infinitely many vertices of degree[1] 0 and infinitely many of degree 1.

(a) Find a substructure $\boldsymbol{A} \subseteq \boldsymbol{B}$ that is isomorphic to $\boldsymbol{B}$ and yet is not an elementary substructure of $\boldsymbol{B}$.

REMARK: Note that $\boldsymbol{A} \cong \boldsymbol{B}$ implies $\boldsymbol{A} \equiv \boldsymbol{B}$. Thus, this is an example of a substructure that is elementarily equivalent to the larger structure and yet isn't an elementary substructure.

(b) Find elementary substructures $\boldsymbol{A}_0, \boldsymbol{A}_1 \preceq \boldsymbol{B}$ with $\boldsymbol{A}_0 \cap \boldsymbol{A}_1 \npreceq \boldsymbol{B}$.

5. Let σ be a signature and show that the following are equivalent:

(I) For any σ-theory T, $T \models \varphi$ implies that there is finite $T_0 \subseteq T$ with $T_0 \models \varphi$.

(II) For any σ-theory T, T not satisfiable (semantically inconsistent) implies that some finite $T_0 \subseteq T$ is not satisfiable.

6. For a fixed signature σ, let $\mathcal{T}_\sigma$ denote the set of all satisfiable maximally complete theories. Denote by $\mathcal{D}_\sigma$ the collection of the subsets of $\mathcal{T}_\sigma$ of the form $\langle\varphi\rangle := \{T \in \mathcal{T} : T \models \varphi\}$, where φ is a σ-sentence. Equip $\mathcal{T}_\sigma$ with the topology generated by $\mathcal{D}_\sigma$.

(a) Show that $\mathcal{D}_\sigma$ is a Boolean is a (Boolean) algebra. In particular, the sets in $\mathcal{D}_\sigma$ are clopen and $\mathcal{D}_\sigma$ is a basis for this topology, making it zero-dimensional[2] Hausdorff.

(b) Prove that the compactness of this space $\mathcal{T}_\sigma$ is equivalent to the statements in Exercise 5.

[1] In an (undirected) graph, the *degree* of a vertex is the number of its neighbors.

[2] A topology is called *zero-dimensional* if it has a basis consisting of clopen sets.

HINT: Use the equivalent statement to compactness that involves closed sets, namely: A topological space is *compact* if and only if every family of closed sets with the finite intersection property[3] has a nonempty intersection.

7. (The Skolem "paradox") Conclude from the Löwenheim–Skolem theorem that any satisfiable σ-theory T has a model of cardinality at most $\max\{|\sigma|, \aleph_0\}$. In particular, if ZFC is satisfiable, then it has a countable model $\boldsymbol{M} := (M, \epsilon^{\boldsymbol{M}})$; without loss of generality, we may assume $M = \mathbb{N}$, so $\epsilon^{\boldsymbol{M}} \subseteq \mathbb{N}^2$.

Let $\varphi(x, y)$ be an (ϵ)-formula expressing a statement that *we read* as `"there is no surjection from` x `to` y`"`. Let $\mathbb{N}^{\boldsymbol{M}}, \mathbb{R}^{\boldsymbol{M}}$ be the elements of the universe $M = \mathbb{N}$ (say, $\mathbb{N}^{\boldsymbol{M}} = 42$, $\mathbb{R}^{\boldsymbol{M}} = 19$) that fulfill (as calculated in $\boldsymbol{M}$) the definitions of what *we read* as `"being the sets of natural numbers and reals"`.

The ZFC axioms and Cantor's theorem ensure that $\boldsymbol{M} \models \varphi(\mathbb{N}^{\boldsymbol{M}}, \mathbb{R}^{\boldsymbol{M}})$, which *we read* as `"reals are uncountable"`. Explain why there is no paradox here.

HINT: Stick with the definitions—avoid philosophy.

REMARK: Wondering whether this is a paradox is analogous to wondering where the missing dollar went in the Missing Dollar Riddle [https://en.wikipedia.org/wiki/Missing_dollar_riddle].

[3] A family $\mathcal{F}$ of sets is said to have the *finite intersection property* if for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$, $\bigcap_{S \in \mathcal{F}_0} S \neq \emptyset$.