Math 570: Mathematical Logic HOMEWORK 2 **Due: Sept 21–22**

1. (a) A σ-formula is called *universal* (resp. *existential*) if it is of the form $\forall x_1 \forall x_2 ... \forall x_n \psi$ (resp. $\exists x_1 \exists x_2 ... \exists x_n \psi$), where ψ is quantifier free. Let $\boldsymbol{A}, \boldsymbol{B}$ be σ-structures with $\boldsymbol{A} \subseteq \boldsymbol{B}$ and let $\varphi(\vec{v})$ be a σ-formula. Show that for any $\vec{a} \in A^n$,

(i) if φ is quantifier free, then $\boldsymbol{A} \models \varphi(\vec{a}) \iff \boldsymbol{B} \models \varphi(\vec{a})$;

(ii) if φ is universal, then $\boldsymbol{B} \models \varphi(\vec{a}) \implies \boldsymbol{A} \models \varphi(\vec{a})$;

(iii) if φ is existential, then $\boldsymbol{A} \models \varphi(\vec{a}) \implies \boldsymbol{B} \models \varphi(\vec{a})$.

(b) Find a sentence that is true in $(\mathbb{N}, <)$ but false in $(\mathbb{Z}, <)$.

2. Let $\vec{v_1}, ..., \vec{v_n} \in \mathbb{Q}^m$. Show that $\{\vec{v_1}, ..., \vec{v_n}\}$ is linearly independent over $\mathbb{Q}$ if and only if it is linearly independent over $\mathbb{R}$.

HINT: Show that linear independence can be expressed by both universal and existential formulas.

3. Determine whether the following are 0-definable; prove your answers unless the question allows guessing.

(a) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.

(b) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.

(c) The set of non-negative numbers in $(\mathbb{Q}, +)$.

(d) The function $\max(x, y)$ in $(\mathbb{R}, <)$.

(e) The function $\text{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.

(f) The element 2 in $(\mathbb{R}, +, \cdot)$.

(g) The set of torsion elements in the group $\mathbb{F}_2 \times \bigoplus_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$, where $\mathbb{F}_2$ is the free group on 2 generators.

HINT: Commutating vs. noncommutating.

(h) (Guess) The set of torsion elements in the group $\mathbb{Z}^2 \times \bigoplus_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$.

(i) The function $n \mapsto n^7$ in $(\mathbb{N}, 0, S, +, \cdot)$.

(j) (Guess) The function $(n, d) \mapsto n^d$ in $(\mathbb{N}, 0, S, +, \cdot)$, where by convention $0^d \mapsto 0$.

HINT: Can you program this function?

4. (a) For $\sigma := (f)$, where f is a unary function symbol, find a σ-sentence, whose only models are infinite σ-structures.

(b) Let $\sigma_{\text{semigp}} := (\cdot)$, where $\cdot$ is a binary function symbol. Find a satisfiable[1] sentence, whose only models are infinite groups under the $\cdot$ operation.

CAUTION: The question isn't asking to find a one-sentence axiomatization for the class of infinite groups: the inclusion is only one way.

HINT: Use the idea of part (a).

[1] *Satisfiable* means it has a model.