Last update: 10:34am, Nov 26.

Math 570: Mathematical Logic HOMEWORK 10 **Due: Nov 30–Dec 1**

1. Prove that the graph of the Ackermann function is primitive recursive. You **may not** use nested recursion.

HINT: Do Dedekind analysis: to verify that the value of the function at (n, x) is w search for a sequence of triples $(i, j, w_{i,j})$ interpreting $w_{i,j}$ as the value of the function at (i, j) and making sure that the triple (n, x, w) appears in this sequence. Find an upper bound for the size of the code of this sequence using w.

REMARK: Although this was assigned earlier, nested recursion (an optional problem) was used in some of your solutions instead of Dedekind analysis. This is being reassigned to ensure mastery of Dedekind analysis.

2. Prove Lemmas 5.32.c and 5.34.

3. (a) Show that we can replace "recursive" by "arithmetical" in the original statement of Gödel's Incompleteness theorem (Theorem 5.2), i.e. prove that if $T \subseteq \mathrm{Th}(\boldsymbol{N})$ is arithmetical, then it is incomplete.

(b) However, show that there exists an arithmetical completion of PA, i.e. there is a maximally complete σ_{arthm}-theory $T \supseteq \mathrm{PA}$ such that $\ulcorner T \urcorner = \{\ulcorner \varphi \urcorner : \varphi \in T\}$ is an arithmetical subset of $\mathbb{N}$. Conclude that we **cannot** replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.

HINT: Mimic the inductive (as opposed to the Zorn's lemma) proof that any theory has a (syntactically) consistent maximal completion.