Math 570: Mathematical Logic HOMEWORK 1 **Due week: Sept 11**

Below, σ denotes a signature. We denote by $\mathrm{Const}(\sigma), \mathrm{Func}(\sigma), \mathrm{Rel}(\sigma)$ the sets of the constant, function, relation symbols in σ, respectively. For each $f \in \mathrm{Func}(\sigma)$ and $R \in \mathrm{Rel}(\sigma)$, $\mathfrak{a}(f)$ and $\mathfrak{a}(R)$ denote their arities. Also, for sets A, B, a set $S \subseteq A \times B$, and $a \in A$, the *fiber of S over a* is the set $S_a := \{b \in B : (a, b) \in S\}$.

1. Let $\boldsymbol{A}$ be a σ-structure and $S \subseteq A$. We say that S *contains the constants of* $\boldsymbol{A}$ if $c^{\boldsymbol{A}} \in S$ for each $c \in \mathrm{Const}(\sigma)$. We also say that S is *closed under the functions of* $\boldsymbol{A}$ if $f^{\boldsymbol{A}}\left(S^{\mathfrak{a}(f)}\right) \subseteq S$ for each $f \in \mathrm{Func}(\sigma)$.

(a) Prove that there is at most one substructure of $\boldsymbol{A}$ with universe S.

(b) Prove that S is the universe of a substructure of $\boldsymbol{A}$ if and only if S contains the constants of $\boldsymbol{A}$ and is closed under the functions of $\boldsymbol{A}$.

(c) Prove that $S_\infty := \bigcup_{n \in \mathbb{N}} S_n$ is the universe of $\langle S \rangle_{\boldsymbol{A}}$, where $S_0 := S \cup \left\{c^{\boldsymbol{A}} : c \in \mathrm{Const}(\sigma)\right\}$ and

$$S_{n+1} = S_n \cup \bigcup_{f \in \mathrm{Func}(\sigma)} f^{\boldsymbol{A}}\left(S_n^{\mathfrak{a}(f)}\right).$$

(d) Conclude that $|\langle S \rangle_{\boldsymbol{A}}| \leq \max\{|S|, \aleph_0, |\sigma|\}$, where $\aleph_0 := |\mathbb{N}|$.

2. Let $\boldsymbol{A}, \boldsymbol{B}$ be σ-structures and let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a σ-homomorphism. Prove:

(a) $h(A)$ is a universe of a substructure of $\boldsymbol{B}$.

(b) If σ does not have any relation symbols, then h is one-to-one if and only if it is a σ-embedding.

REMARK: That's why this happens with groups and rings, but not with graphs or orderings.

3. A σ-structure is called *rigid* if it has no σ-automorphisms (i.e. isomorphism with itself) other than the identity. Show that the structures $(\mathbb{N}, 0, S, +, \cdot)$ and $(\mathbb{Q}, 0, 1, +, \cdot)$ are rigid, where the symbols have their usual interpretation.

4. A σ-structure $\boldsymbol{A}$ is called *ultrahomogeneous* if any σ-isomorphism between two finitely generated substructures extends to an automorphism of the whole structure $\boldsymbol{A}$, i.e. if $\boldsymbol{B}, \boldsymbol{C}$ are finitely generated substructures of $\boldsymbol{A}$ and $h : \boldsymbol{B} \to \boldsymbol{C}$ is a σ-isomorphism, then there is a σ-automorphism $\bar{h}$ of $\boldsymbol{A}$ with $\bar{h} \supseteq h$.

(a) Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. The same argument should prove that $(\mathbb{R}, <)$ is also ultrahomogeneous.

(b) Let $\sigma_{\mathrm{gr}} := (E)$ be the signature for graphs and let $\mathbf{G}$ be a graph (i.e. a σ_{gr}-structure) that the undirected[1] bi-infinite path[2] (looks like the $\mathbb{Z}$-line). Is $\mathbf{G}$ ultrahomogeneous? Prove your answer.

[1]By an *undirected* graph we mean that E is a symmetric relation.

[2]More precisely, it is the unique infinite connected 2-regular undirected graph.

5. For σ-structures $\boldsymbol{A}, \boldsymbol{B}$, we call $\boldsymbol{A}$ and $\boldsymbol{B}$ *elementarily equivalent*, written $\boldsymbol{A} \equiv \boldsymbol{B}$, if for every σ-sentence φ, $\boldsymbol{A} \models \varphi \iff \boldsymbol{B} \models \varphi$.

(a) For a fixed finite group $\mathbf{G}$ in the signature $\sigma_{\text{gp}} := (1, \cdot, ()^{-1})$, prove that there is a σ_{gp}-sentence φ such that for any σ_{gp}-structure $\boldsymbol{B}$,

$$\boldsymbol{B} \models \varphi \iff \boldsymbol{B} \cong \mathbf{G}.$$

HINT: A group is determined by its multiplication table. Use $\exists$ to be able to talk about the elements of G.

(b) More generally, let σ be a finite signature and $\boldsymbol{A}$ be a finite σ-structure. Show that there is a σ-sentence φ such that for any σ-structure $\boldsymbol{B}$,

$$\boldsymbol{B} \models \varphi \iff \boldsymbol{B} \cong \boldsymbol{A}.$$

In particular,

$$\boldsymbol{B} \equiv \boldsymbol{A} \iff \boldsymbol{B} \cong \boldsymbol{A}.$$

REMARK: Part (a) is a special case of part (b), so you don't have to do (a) if you do (b) successfully, but (a) might give you an idea of how to approach (b).

Definition 1. For a set A and a collection $\mathcal{S} \subseteq \bigcup_{n \geq 1} \mathscr{P}(A^n)$, put $\mathcal{S}_n := \mathcal{S} \cap \mathscr{P}(A^n)$. For a set $P \subseteq A$, call $\mathcal{S}$ *P-constructively closed* if:

(i) *Boolean algebra*: Each $\mathcal{S}_n$ is a Boolean algebra, i.e. contains $\emptyset$ and A^n and is closed under complements and (finite) unions.

(ii) *Symmetry*: Each $\mathcal{S}_n$ is symmetric, i.e. closed under any permutation of coordinates.

(iii) *Projections*: The projection onto the first n coordinates of any set in $\mathcal{S}_{n+1}$ is in $\mathcal{S}_n$, i.e. $\forall S \in \mathcal{S}_{n+1}$, $\text{proj}_{(0,1,\ldots,n-1)}(S) \in \mathcal{S}_n$.

(iv) *Lifts*: The Cartesian product of A with any set in $\mathcal{S}_n$ is in $\mathcal{S}_{n+1}$, i.e. $S \in \mathcal{S}_n \Rightarrow A \times S \in \mathcal{S}_{n+1}$.

(v) *P-fibers*: The fiber over any element $p \in P$ of any set in $\mathcal{S}_{1+n}$ is in $\mathcal{S}_n$, i.e. $S \in \mathcal{S}_{1+n} \Rightarrow S_p \in \mathcal{S}_n$.

6. For any σ-structure $\boldsymbol{A}$ and $P \subseteq A$, prove that the collection $\mathcal{D}^{\boldsymbol{A}}(P)$ of P-definable sets of $\boldsymbol{A}$ is the smallest P-constructively closed collection containing the constant singletons $\{c^{\boldsymbol{A}}\}$, the graphs $\text{Graph}(f^{\boldsymbol{A}})$ of the functions, and the relations $= \subseteq A^2$ and $R^{\boldsymbol{A}}$, for all $c, f, R \in \sigma$.

HINT: To show that $\mathcal{D}^{\boldsymbol{A}}(P)$ is the *smallest* such collection, induct on the construction of σ-formulas.