Math 347H: Fundamental Math (H) HOMEWORK 7 **Due date: Nov 9 (Thu)**

1. Prove that there is no $q \in \mathbb{Q}$ with $q^2 = 3$. Informally speaking, the question asks to prove that $\sqrt{3}$ is not rational.

2. Equivalence relations induced by functions.

(a) For a function $f : X \to Y$, define a binary relation E_f on X by

$$x_0 E_f x_1 :\Leftrightarrow f(x_0) = f(x_1).$$

Prove that E_f is an equivalence relation. We call it the equivalence relation induced by f.

(b) Let $E_{\mathbb{Z}}$ be the binary relation on $\mathbb{R}$ defined by $x E_{\mathbb{Z}} y :\Leftrightarrow x - y \in \mathbb{Z}$. Prove that $E_{\mathbb{Z}}$ is an equivalence relation and find a function $f : \mathbb{R} \to [0,1)$ such that $E_{\mathbb{Z}} = E_f$.

(c) More generally, for any equivalence relation E on a set X, find a set Y and a function $f : X \to Y$ such that $E = E_f$.

HINT: Quotient by E.

3. Let $F(\mathbb{Q})$ denote the set of all functions $\mathbb{Q} \to \mathbb{R}$, so each element $f \in F(\mathbb{Q})$ is a function from $\mathbb{Q}$ to $\mathbb{R}$. Define a function $\delta_0 : F(\mathbb{Q}) \to \mathbb{R}$ by mapping each $f \in F(\mathbb{Q})$ to its value at 0, i.e. $\delta_0(f) := f(0)$. This function is called the *Dirac distribution* at 0.

(a) Prove that δ_0 is surjective.

(b) Explicitly define two distinct right-inverses for δ_0.

(c) Letting $M_<(\mathbb{Q})$ be the subset of $F(\mathbb{Q})$ of all strictly increasing functions, determine the sets $\delta_0\big(M_<(\mathbb{Q})\big)$ and $\delta_0\big(M_<(\mathbb{Q})^c\big)$.

(d) Determine the set $\delta_0^{-1}(\mathbb{Z})$.

4. Let $F(\mathbb{R})$ denote the set of all functions $\mathbb{R} \to \mathbb{R}$. The composition $f \circ g$ of two functions $f, g \in F(\mathbb{R})$ is a binary operation on $F(\mathbb{R})$. Determine whether

(a) $\circ$ is associative;

(b) $\circ$ is commutative;

(c) there is a $\circ$-identity;

(d) every $f \in F(\mathbb{R})$ has a $\circ$-inverse.

Prove each of your answers. If an answer is negative, provide an explicit counter-example.

5. For sets A, B, recall that we write $A \cong B$ to mean that there is a bijection $A \xrightarrow{\sim} B$; in this case, we say that A and B are *equinumerous*. Prove that the following sets are equinumerous with $\mathbb{N}$.

(a) $\mathbb{N}^+$.

(b) The set of all odd numbers natural numbers;

(c) $\mathbb{Z}$;

(d) The set of all integers divisible by 6;

(e) $\mathbb{N}^2$;

(f) $\mathbb{N}^7$.