Math 347H: Fundamental Math (H) HOMEWORK 10 **Due date: Dec 7 (Thu)**

1. For the set $A := \left\{x \in \mathbb{Q} \setminus \{0\} : x < 1 + \frac{1}{x}\right\}$, let $\phi := \sup A$. Determine #dbp(ϕ) and the first 4 digits of ϕ using the method given in the proof of the order-completeness of $\mathbb{R}$.

REMARK: This number ϕ is known as the *golden ratio*.

2. Let $A, B \subseteq \mathbb{R}$ be nonempty sets bounded above. Prove that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

3. For each set S below, find $\inf S$ and $\sup S$ (within $\mathbb{R}$). Prove your answers.

(a) $S = \left\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}^+\right\}$;

(b) $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}^+\right\}$.

4. Which of the following functions $d : X \times X \to [0, \infty)$ is a metric on X? Prove your answers.

(a) $X := \mathbb{R}$ and $d(x,y) := \begin{cases} |x-y| & \text{if } |x-y| \leq 1 \\ 1 & \text{otherwise} \end{cases}$

(b) $X := \mathbb{R}$ and $d(x,y) := (x-y)^2$

(c) $X := (0,1)$ and $d(x,y) := |x-y| + |\frac{1}{x} - \frac{1}{y}|$

(d) $X := \mathbb{R}^n$ and $d(x,y) := \max_{i<n} |x_i - y_i|$, where $x := (x_0, \ldots, x_{n-1})$, $y := (y_0, \ldots, y_{n-1})$.

5. For fixed $n \in \mathbb{N}^+$ and a set $A \neq \emptyset$, let $X = A^n$ and think of the elements of X as words of length n in the alphabet A. Define $d_H : X \times X \to [0, \infty)$ by setting $d_H(x,y)$ to be the total number of indices $i < n$ at which x and y differ, i.e.

$$d(x,y) := |\{i \in \{0, \ldots, n-1\} : x_i \neq y_i\}|.$$

Prove that this is a metric on X.

REMARK: d_H is called *Hamming metric*.

6. Let (X,d) be a metric space and prove that for any $x, y \in X$, the following are equivalent:

(1) (Algebraic statement, no room to wiggle) $x = y$

(2) (Analytic statement, ε of wiggle room) For each $\varepsilon > 0$, $d(x,y) \leq \varepsilon$.