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MATH 571: MODEL THEORY

PROBLEMS

1. Definability

1.1. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $P \subseteq A$. Show that any automorphism $h : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ that fixes P pointwise (i.e. for every $p \in P$, $h(p) = p$) must fix every P-definable set $D \subseteq A^n$ setwise (i.e. $h(D) = D$, where, as usual, $h(\vec{d}) = (h(d_1), ..., h(d_n))$ for $\vec{d} \in D$).

1.2. Determine whether the following are 0-definable. You may choose to do only 5 parts of this exercise.

(a) The set $\mathbb{N}$ in $(\mathbb{Z}, 0, +)$.
(b) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.

Hint: You need a nontrivial fact from elementary number theory.
(c) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.
(d) The set $\pi\mathbb{Z}$ in $(\mathbb{C}, 0, 1, +, \cdot, \exp)$, where $\exp : \mathbb{C} \to \mathbb{C}$ the usual exponentiation function given by $z \mapsto e^z$.
(e) The set of positive numbers in $(\mathbb{R}, <)$.
(f) The function $\max(x, y)$ in $(\mathbb{R}, <)$.
(g) The function $\text{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.
(h) The element 2 in $(\mathbb{R}, +, \cdot)$.
(i) The relation $d(x, y) = 2$ in an undirected graph (with no loops) (Γ, E), where $d(x, y)$ denotes the edge distance function.

Hint: Use the previous problem to prove the negative answers.

1.3. Let $\mathcal{L}_{\text{gp}} := (\cdot, ()^{-1}, 1)$ be the *language of groups* and by a *group* we mean a structure in this language satisfying the usual group axioms.

(a) Is the set of torsion elements 0-definable in the direct sum $G := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \cdots$? What about $G := Z$?

(b) Prove that there is no $\mathcal{L}_{\text{gp}}$-formula $\varphi(x)$ such that for *every* group G, $\varphi(x)$ defines the set of all torsion elements of G.

1.4. Use the compactness theorem to show that the connectedness relation

$$P(x, y) \iff x \text{ and } y \text{ are connected}$$

is not definable in any disconnected acyclic 2-regular[1] (undirected) graph $\mathcal{G} := (V; E)$, in other words, a graph that consists of at least two disjoint bi-infinite paths.

Hint: Let $\varphi(x, y, \vec{z})$ be a hypothetical $\mathcal{L}_{\text{gr}} := (E)$-formula such that for some parameters $\vec{p} \subseteq V$, $\varphi(x, y, \vec{p})$ defines the relation $P(x, y)$ above. Let A be a connected component of $\mathcal{G}$. Use two new constant symbols and the Compactness theorem to obtain an extension $\bar{\mathcal{G}}$ of $\mathcal{G}$ containing (at least) two new connected components B, C such that φ holds between the

[1] A graph is *k-regular* if each vertex has exactly k neighbors.

elements of B and A, but fails between those of C and A. Get a contradiction by swapping B and C.

2. Boolean algebras and Stone spaces

2.1. Let A be a Boolean algebra and let a, b range over its elements. Prove that

(a) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$;

(b) $a \vee b = b \Leftrightarrow a \wedge b = a$;

(c) $a \leq b :\Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$ defines a partial order on A;

(d) a' is the unique element b with $a \vee b = 1$ and $a \wedge b = 0$;

Hint: Show that $a' \leq b$ and $a' \geq b$.

2.2. Let A be a Boolean algebra and $S \subseteq A$ be such that the meet (i.e. $\bigwedge$) of any finite subset of S is not equal to 0. Prove that S can be extended to a proper filter, and hence to an ultrafilter, of A.

2.3. Let A be a Boolean algebra, F a filter of A and $a \in A$. Carefully prove the following.

(a) $F \wedge a := \{b \in A : b \geq f \cap a \text{ for some } f \in F\}$ is the filter generated by F.

(b) If $a' \notin F$ then $F \wedge a$ is a proper filter.

(c) $F = \bigcap \{\alpha \in \mathrm{St}(A) : \alpha \supseteq F\}$.

(d) $a \mapsto [a]$ is a Boolean algebra embedding of A into the Boolean algebra of clopen sets of $\mathrm{St}(A)$.

For an algebra A and $S \subseteq A$, let $\langle S \rangle_{\mathrm{alg}}$ (resp. $\langle S \rangle_{\mathrm{lttc}}$) denote the closure of S under the operations of $\vee, ()'$ (resp. $\vee, \wedge$) and call it the *algebra* (resp. *lattice*) generated by S.

2.4. Write down explicitly what the set $\langle S \rangle_{\mathrm{lttc}}$ is for any subset S of a given algebra A, i.e. figure out the form of the elements of $\langle S \rangle_{\mathrm{lttc}}$ in terms of those of S and $\vee, \wedge$.

For $S \subseteq A$ and $a \in A$, say that S *separates points across* a if for any ultrafilters $\alpha \in [a]$ and $\beta \notin [a]$, there is $s \in S$ with $s \in \alpha$ and $s \notin \beta$ (equivalently, $\alpha \in [s]$ but $\beta \notin [s]$). S is said to *separate points* if it separates across every $a \in A$.

Theorem 2.5. *Let A be a Boolean algebra, let $S \subseteq A$ and $a \in A$. $a \in \langle S \rangle_{\mathrm{lttc}}$ if and only if S separates points across a.*

In class, we proved the equivalence of the first two statements in the following.

Corollary 2.6. *Let A be a Boolean algebra and let $S \subseteq A$. The following are equivalent.*

(1) $\langle S \rangle_{\mathrm{alg}} = A$.
(2) *S separates points.*
(3) $\langle S \rangle_{\mathrm{lttc}} = A$.

2.7. Prove Theorem 2.5 and conclude Corollary 2.6.

Hint for $\Leftarrow$ of Theorem 2.5: It is enough to show that every point $p \in [a]$ is contained in $[s] \subseteq [a]$ for some $s \in S$. To get such an s, try to separate p from every point of $[a']$.

3. Types

3.1. Let Σ be an $\mathcal{L}$-theory and let $\mathcal{A}_{\vec{x}}^{\mathcal{L}}(\Sigma)$ denote the algebra of all $\mathcal{L}$-formulas modulo the equivalence in Σ, i.e.

$$\mathcal{A}_{\vec{x}}^{\mathcal{L}}(\Sigma) := \text{Formulas}(\mathcal{L}, \vec{x}) / \sim_\Sigma .$$

For an $\mathcal{L}$-formula $\varphi(\vec{x})$, let $[\varphi(\vec{x})] \sim_\Sigma$ denote its $\sim_\Sigma$-equivalence class.

Also recall that for an $\mathcal{L}$-formula $\varphi(\vec{x})$ and an $\mathcal{L}$-structure $\mathcal{M}$, we denote by $[\varphi(\vec{x})]_{\mathcal{M}}$ the subset of $M^{|\vec{x}|}$ defined by $\varphi(\vec{x})$. Lastly, denote by $\mathcal{D}_{\vec{x}}(\mathcal{M})$ the collection of all definable (in $\mathcal{M}$) subsets of $M^{|\vec{x}|}$.

(a) For a model $\mathcal{M} \models \Sigma$, the map $[\varphi(\vec{x})]_{\sim_\Sigma} \mapsto [\varphi(\vec{x})]_{\mathcal{M}}$ is a well-defined Boolean algebra homomorphism $\mathcal{A}_{\vec{x}}^{\mathcal{L}}(\Sigma) \to \mathcal{D}_{\vec{x}}(\mathcal{M})$.

(b) The map $p(\vec{x}) \mapsto \{[\varphi(\vec{x})]_{\sim_\Sigma} : \varphi(\vec{x}) \in p(\vec{x})\}$ is a homeomorphism $S_{\vec{x}}(\Sigma) \xrightarrow{\sim} \text{St}\big(\mathcal{A}_{\vec{x}}^{\mathcal{L}}(\Sigma)\big)$.

3.2. Let Σ be an $\mathcal{L}$-theory and, for an $\mathcal{L}$-formula $\varphi(\vec{x})$, let $[\varphi(\vec{x})]_{S(\Sigma)}$ denote the clopen subset of the type space $S_{\vec{x}}(\Sigma)$ defined by $\varphi(\vec{x})$, i.e. the collection of all $\vec{x}$-types containing $\varphi(\vec{x})$. Similarly, for a partial type $\Phi(\vec{x})$, put

$$\big[\Phi(\vec{x})\big]_{S(\Sigma)} := \bigcap_{\varphi(\vec{x}) \in \Phi(\vec{x})} [\varphi(\vec{x})]_{S(\Sigma)}.$$

(a) Verify the equivalence of the following statements:
  (i) $[\varphi(\vec{x})]_{S(\Sigma)} \neq \varnothing$,
  (ii) $\varphi(\vec{x}) \nsim_\Sigma \bot$, i.e. $\Sigma \nvDash (\varphi(\vec{x}) \leftrightarrow \bot)$,
  (iii) $\varphi(\vec{x})$ is Σ-*realizable*, i.e. $\Sigma \cup \{\varphi(\vec{x})\}$ is satisfiable (as an $\mathcal{L}(\vec{x})$-theory).

(b) Observe that $\big[\Phi(\vec{x})\big]_{S(\Sigma)}$ is a closed subset of $S_{\vec{x}}(\Sigma)$ and that every closed subset is of this form.

(c) Prove that $\big[\Phi(\vec{x})\big]_{S(\Sigma)}$ has empty interior if and only if Σ *locally omits* $\Phi(\vec{x})$, that is: for any Σ-realizable formula $\psi(\vec{x})$, there is $\varphi(\vec{x}) \in \Phi(\vec{x})$ such that $\psi(\vec{x}) \cap \neg\varphi(\vec{x})$.

3.3. In class we proved that if Σ locally omits a partial type $\Phi(x)$ then there is a model of Σ that omits $\Phi(x)$. Modify this proof to turn it into a proof of the full version of the Omitting Types theorem, i.e. for a countable collection $\big\{\Phi_n(\overrightarrow{x_n})\big\}_{n \in \mathbb{N}}$ of partial types locally omitted by Σ.

4. Partial isomorphisms and elementary maps

4.1. Prove that the limit/union of any chain $(\mathcal{M}_i)_{i \in I}$ of $\mathcal{L}$-structures is a well-defined $\mathcal{L}$-structure extending each $\mathcal{M}_i$.

4.2. Let $\mathcal{N}$ be an $\mathcal{L}$-structure, $\mathcal{M}_0, \mathcal{M}_1 \subseteq \mathcal{N}$ and $M := M_0 \cap M_1$.

(a) Prove that M is an underlying set of a substructure, which we denote by $\mathcal{M}$. Thus, the intersection of two substructures is a substructure.

(b) Construct your own example showing that even if $\mathcal{M}_0, \mathcal{M}_1 \preceq \mathcal{N}$, $\mathcal{M}$ need not be elementary.

(c) However, prove that if $\mathcal{M}_0, \mathcal{M}_1 \preceq \mathcal{N}$, then the identity map $h := \mathrm{id}_M$ is a partial **elementary** map $\mathcal{M}_0 \rightharpoonup \mathcal{M}_1$ (with domain M).

(d) Prove that being an elementary substructure is transitive in every way:

(i) If $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{N}$ then $\mathcal{M}_0 \preceq \mathcal{N}$;

(ii) If $\mathcal{M}_0 \subseteq \mathcal{M}_1 \preceq \mathcal{N}$ and $\mathcal{M}_0 \preceq \mathcal{N}$ then $\mathcal{M}_0 \preceq \mathcal{M}_1$.

4.3. (By Anton Bernshteyn) Let $\mathcal{B} = (B, E)$ be a countable undirected graph, each of whose vertices have degree at most 1. Suppose further that $\mathcal{B}$ has infinitely many vertices of degree 0 and infinitely many of degree 1. Find a substructure $\mathcal{A} \subseteq \mathcal{B}$ that is isomorphic to $\mathcal{B}$ and yet is not an elementary substructure of $\mathcal{B}$. For simplicity, you may assume that $\mathcal{B}$ is countable.

4.4. Let $\mathcal{L}_S := (0, S)$ be a language, where 0 is a constant symbol and S a unary function symbol. Let T_S be the $\mathcal{L}_S$-theory consisting of the following axioms:

- Zero has no predecessor: $\forall x (S(x) \neq 0)$.
- The successor function is one-to-one: $\forall x \forall y (S(x) = S(y) \to x = y)$.
- Any nonzero number is a successor of something: $\forall x (x \neq 0 \to \exists y (x = S(y)))$.
- For all $n \in \mathbb{N}$, there are no n-cycles: $\forall x (S^n(x) \neq x)$, where S^n stands for the n-fold composition of S.

For any $\mathcal{M}_0, \mathcal{M}_1 \models T_S$, determine exactly which partial isomorphisms $f : \mathcal{M}_0 \rightharpoonup \mathcal{M}_1$ admit a back-and-forth system $\mathcal{F} : \mathcal{M}_0 \leftrightarrows \mathcal{M}_1$ containing f.

5. Classifying the models of a theory

5.1. Let Σ_0, Σ_1 be $\mathcal{L}$-theories. Prove that if Σ_0 complete and $\Sigma_0 \cup \Sigma_1$ is consistent, then $\Sigma_0 \Rightarrow \Sigma_1$, i.e. $\Sigma_0 \models \varphi$, for every $\varphi \in \Sigma_1$. Conclude that if both Σ_0, Σ_1 are complete and $\Sigma_0 \cup \Sigma_1$ is consistent, then $\Sigma_0 \equiv \Sigma_1$.

5.2. Let $\mathcal{L}$ be a finite language. Prove that the theory of a finite $\mathcal{L}$-structure has exactly one model (up to isomorphism).

5.3. Let $\mathfrak{C}$ be a class (a set) of $\mathcal{L}$-structures. Define the *theory* $\mathrm{Th}(\mathfrak{C})$ and the *asymptotic theory* $\mathrm{Th}_a(\mathfrak{C})$ of $\mathfrak{C}$ as follows: for every $\mathcal{L}$-sentence φ,

$$\varphi \in \mathrm{Th}(\mathfrak{C}) :\Leftrightarrow \forall \mathcal{M} \in \mathfrak{C} \;\; \mathcal{M} \models \varphi,$$
$$\varphi \in \mathrm{Th}_a(\mathfrak{C}) :\Leftrightarrow \forall^\infty \mathcal{M} \in \mathfrak{C} \;\; \mathcal{M} \models \varphi,$$

where $\forall^\infty$ means "for all but finitely many".

(a) Let $\mathfrak{C}$ be an infinite class of finite structures that contains only finitely many structures of cardinality n, for each $n \in \mathbb{N}$. Prove that the models of $\mathrm{Th}_a(\mathfrak{C})$ are exactly the infinite models of $\mathrm{Th}(\mathfrak{C})$.

(b) Let $\mathcal{L} := \mathcal{L}_\varnothing$ (the empty language) and suppose $\mathfrak{C}$ contains arbitrarily large finite structures. What theories are $\mathrm{Th}(\mathfrak{C})$ and $\mathrm{Th}_a(\mathfrak{C})$ equivalent to?

5.4. Let $\mathcal{L}_{\mathrm{gph}} := (E)$ (the language of graphs) and let $\mathfrak{C} := \{\mathcal{G}_n\}_{n\geq 2}$, where $\mathcal{G}_n$ is the undirected chain of n vertices; more precisely $\mathcal{G}_n := (G_n, E^{\mathcal{G}_n})$, where $G_n := \{1, 2, \dots, n\}$ and $uE^{\mathcal{G}_n}v \Leftrightarrow |u - v| = 1$, for $u, v \in G_n$.

(a) Exhibit a nontrivial $\mathcal{L}_{\mathrm{gph}}$-sentence from $\mathrm{Th}(\mathfrak{C})$, i.e. a sentence that doesn't follow from the axioms of the first order logic or from axioms of undirected graphs with no loops[2].

(b) List all infinite models of $\mathrm{Th}(\mathfrak{C})$.

(c) List all finite models of $\mathrm{Th}(\mathfrak{C})$.

5.5. Recall that we showed in class that DLO is $\aleph_0$-categorical. Take a sequence fresh constant symbols $(c_i)_{i\in\mathbb{N}}$.

(a) Show that for every $n \in \mathbb{N}$, the theory

$$\mathrm{DLO}_n = \mathrm{DLO} \cup \{c_i < c_{i+1} : i < n\}$$

is still $\aleph_0$-categorical as a theory in the language $(<, \{c_i : i \leq n\})$. Conclude that DLO_n is complete.

(b) Conclude that the theory

$$\mathrm{DLO}_\infty = \bigcup_{n\in\mathbb{N}} \mathrm{DLO}_n$$

is complete as a theory in the language $(<, \{c_i : i \in \mathbb{N}\})$.

(c) Yet, show that DLO_∞ has exactly three countable models up to isomorphisms; in particular, DLO_∞ is not $\aleph_0$-categorical.

6. Quantifier elimination

First, we recall some criteria for q.e. proved or mentioned in class.

Theorem 6.1 (Q.e. via back-and-forth)**.** *Let Σ be an $\mathcal{L}$-theory and suppose that for every q.f. type $q(\vec{x})$ and any extensions $p_0(\vec{x}), p_1(\vec{x}) \in S^{\mathcal{L}}_{\vec{x}}(\Sigma)$ of $q(\vec{x})$, there are models $\mathcal{M}_0, \mathcal{M}_1 \models \Sigma$ realizing $p_0(\vec{x})$ and $p_1(\vec{x})$, respectively, by tuples $\vec{a} \in M_0, \vec{b} \in M_1$, and a back-and-forth system $\mathcal{F} : \mathcal{M}_0 \leftrightarrows \mathcal{M}_1$ containing the map $\vec{a} \rightrightarrows \vec{b}$. Then, Σ admits q.e.*

For an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subseteq M$, recall that $\mathcal{L}(A)$ denotes the expansion of the language $\mathcal{L}$ with fresh constants, one for each element of A; we let $\mathcal{M}(A)$ denote the expansion of $\mathcal{M}$ to an $\mathcal{L}(A)$-structure, where the constants in A are interpreted by themselves.

For an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subseteq M$, let $\mathrm{ElDiag}_{\mathcal{M}}(A)$ (resp. $\mathrm{Diag}_{\mathcal{M}}(A)$) denote all (resp. quantifier free) sentences φ in the language $\mathcal{L}(A)$ that are true about the elements of A, i.e. $\mathcal{M}(A) \models \varphi$. Call the set $\mathrm{ElDiag}_{\mathcal{M}}(A)$ (resp. $\mathrm{Diag}_{\mathcal{M}}(A)$) the *elementary diagram* (resp. *diagram* or *q.f. diagram*) of A in $\mathcal{M}$.

Note that for a finite tuple $\vec{a} \in M$, the diagram of $\vec{a}$ is exactly its q.f. type.

Finally, say that an $\mathcal{L}$-theory Σ *decides* an $\mathcal{L}$-sentence φ if either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$.

[2]In a graph, a *loop* is an edge from a vertex to itself.

Theorem 6.2. *For an $\mathcal{L}$-theory Σ and an $\mathcal{L}$-formula $\varphi(\vec{x})$, the following are equivalent:*

(1) *$\varphi(\vec{x})$ is Σ-equivalent to a q.f. $\mathcal{L}$-formula $\psi(\vec{x})$.*

(2) *For every model $\mathcal{M} \vDash \Sigma$ and any $\vec{a} \in M^{|\vec{x}|}$, $\Sigma \cup \mathrm{Diag}_{\mathcal{M}}(\vec{a})$ decides $\varphi(\vec{a})$.*

(3) *For every q.f. type $q(\vec{x})$ of Σ, $\Sigma \cup q(\vec{x})$ decides $\varphi(\vec{x})$.*

The last theorem allows for rephrasing q.e. in terms of completeness.

Corollary 6.3. *For an $\mathcal{L}$-theory Σ, the following are equivalent:*

(1) *Σ admits q.e.*

(2) *For every model $\mathcal{M} \vDash \Sigma$ and any $\vec{a} \in M$, $\Sigma \cup \mathrm{Diag}_{\mathcal{M}}(\vec{a})$ is a complete $\mathcal{L}(\vec{a})$-theory.*

(3) *For every q.f. type $q(\vec{x})$ of Σ, $\Sigma \cup q(\vec{x})$ is a complete $\mathcal{L}(\vec{x})$-theory.*

6.4. Determine all q.f. 0-dimensional (i.e. no variables) types of ACF; more precisely, list all of the elements of $S_0(\mathrm{ACF})$ and point out exactly which elements are isolated (as points in the topological space $S_0(\mathrm{ACF})$).

6.5. Prove Theorem 6.2 and conclude Corollary 6.3. Realize that one can use categoricity to prove q.e.

HINT: Deduce Theorem 6.2 from Theorem 2.5.

6.6. Prove q.e. for the following theories. You may use any criterion for two of the theories, but prove it by hand (syntactically) for one of them.

(a) DLO,

(b) the theory SUCC of the successor, that is: the axiomatization of $\mathrm{Th}(\mathbb{N}, 0, S)$ that we defined in class,

(c) the theory VEC_F of vector spaces over a fixed field F.

6.7. Let K be an algebraically closed field and $F \subseteq K$ a subfield. Prove that the continuous injection $S_{\vec{x}}(\mathrm{ACF}/F) \to \mathrm{Spec}(F[\vec{x}])$ is surjective.

HINT: Given $J \in \mathrm{Spec}(F[\vec{x}])$, one has to show that the unique candidate for its preimage is actually a realizable type. Strong Nullstellensatz is what provides realizations of these kinds of types, but it only applies to ideals of $K[\vec{x}]$ and not $F[\vec{x}]$. Use without proof that J lifts to a prime ideal of $K[\vec{x}]$, i.e. there is $J' \in \mathrm{Spec}(K[\vec{x}])$ such that $J' \cap F[\vec{x}] = J$.

7. Model completeness

Definition 7.1. A theory Σ is called *model complete* if for any models $\mathcal{M}, \mathcal{N} \vDash \Sigma$, $\mathcal{M} \subseteq \mathcal{N}$ implies $\mathcal{M} \preceq \mathcal{N}$.

Theorem 7.2. *For an $\mathcal{L}$-theory $T\Sigma$, the following are equivalent:*

(1) *Σ is model-complete.*

(2) *For every model* $\mathcal{M} \vDash \Sigma$, $\Sigma \cup \mathrm{Diag}_{\mathcal{M}}(M)$ *is a complete* $\mathcal{L}(M)$*-theory.*

(3) *Every* $\mathcal{L}$*-formula* $\varphi(\vec{x})$ *is* Σ*-equivalent to a universal*[3] *formula.*

(4) *Every* $\mathcal{L}$*-formula* $\varphi(\vec{x})$ *is* Σ*-equivalent to an existential*[3] *formula.*

7.3. Prove Theorem 7.2. Compare it with Corollary 6.3.

Hint: To prove (2)⇒(3) of Theorem 7.2, use (2)⇒(3) of Corollary 2.6.

7.4. Let $\mathcal{L}_{\text{gph}} \coloneqq (E)$ be the language of graphs, i.e. E is a binary relation symbol.

(a) Write down an explicit axiomatization T for the class of undirected graphs with no loops, whose connected components are *bi-infinite chains*, i.e. acyclic graphs with the degree of each vertex being 2.

(b) Show that T is complete.

(c) Conclude yet again (for the last time, I promise) that the relation $R(v,u)$ of being in the same connected component is not 0-definable in any disconnected model of T.

Hint: Let $\mathcal{M}$ be such a model, so $\mathcal{M} \vDash \exists x \exists y \neg R(x,y)$, hence $\mathbf{Z} \vDash \exists x \exists y \neg R(x,y)$, where $\mathbf{Z}$ is the connected model. Therefore, $\mathbf{Z} \vDash \exists x \exists y \big[\neg R(x,y) \wedge \mathrm{dist}_{\leq d}(x,y) \big]$, for some $d \in \mathbb{N}$. But now $\mathcal{M}$ must satisfy the latter sentence too, which is a contradiction.

(d) Show that for any $\mathcal{M} \vDash T$ and $a, b \in M$, there is an automorphism g of $\mathcal{M}$ with $g(a) = b$.

(e) For $\mathcal{M} \vDash T$, exactly which subsets of M are 0-definable in $\mathcal{M}$?

(f) Finally, prove that T is model-complete, but does not admit q.e.

Remark: You may assume that our first-order language always includes the 0-ary relation symbols $\top$ and $\bot$ for truth and falsehood, respectively, so $\mathcal{L}_{\text{gph}}$ not having a constant symbol isn't the reason why T doesn't admit q.e.

8. Saturation

Theorem 8.1. *Let* $\mathcal{L}$ *be a countable language and* α *a nonprincipal ultrafilter on* $\mathbb{N}$*. The ultraproduct* $\mathcal{M}_\infty$ *over* α *of any sequence* $(\mathcal{M}_i)_{i \in \mathbb{N}}$ *of* $\mathcal{L}$*-structures is countably saturated.*

8.2. Follow the steps below to prove Theorem 8.1.

(i) Fix a countable $A \subseteq M_\infty$ and argue that it is enough to prove that $\bigcap_{n \in \mathbb{N}} B^{(n)} \neq \varnothing$ for some countable FIP collection $\{B^{(n)}\}_{n \in \mathbb{N}}$ of A-definable subsets of M_∞.

(ii) Show that each $B^{(n)}$ itself can be represented as an ultraproduct of sets defined by the same formula in respective models $\mathcal{M}_i$; more explicitly,

$$B^{(n)} \coloneqq \left[\prod_{i \in \mathbb{N}} B_i^{(n)} \right]_\alpha .$$

[3] A formula is called *universal* (resp. *existential*) if it is of the form $\forall \vec{y} \psi(\vec{x}, \vec{y})$ (resp. $\exists \vec{y} \psi(\vec{x}, \vec{y})$), where ψ is a q.f. formula.

(This part is intentionally left somewhat vague and interpreting it is part of the question.)

(iii) Prove that for each $N \in \mathbb{N}$, we have $(\forall^\alpha i \in \mathbb{N}) \bigcap_{n \leq N} B_i^{(n)} \neq \varnothing$.

(iv) For each $i \in \mathbb{N}$, let N_i denote the largest natural number $\leq i$ such that $\bigcap_{n \leq N_i} B^{(n)} \neq \varnothing$. Use this to define the i^{th} coordinate of the hypothetical point $x \in \bigcap_{n \in \mathbb{N}} B^{(n)}$.

Theorem 8.3 (Blum's q.e. criterion)**.** *An $\mathcal{L}$-theory T admits q.e. if and only if for all models $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{N}$ being $|M|^+$-saturated, and for each substructure $\mathcal{A} \subseteq \mathcal{M}$, every embedding $\mathcal{A} \hookrightarrow \mathcal{N}$ extends to an embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.*

8.4. Prove Theorem 8.3

Hint: For $\Rightarrow$, enumerate M and build the embedding $\mathcal{M} \hookrightarrow \mathcal{N}$ by transfinite induction. For $\Leftarrow$, it is enough to fix a q.f. formula $\varphi(\vec{x}, y)$ and show that the formula $\exists y \varphi(\vec{x}, y)$ is T-equivalent to a q.f. formula. Do this using Theorem 6.2.

8.5. Recall that an $\mathcal{L}$-structure $\mathcal{M}$ is called *saturated* if it is $|M|$-saturated. Prove that any two saturated elementarily equivalent equinumerous[4] $\mathcal{L}$-structures are isomorphic.

8.6. Show that if an $\mathcal{L}$-structure $\mathcal{M}$ is κ-saturated, then all definable (with parameters) sets (in all dimensions) are either finite or of cardinality at least κ.

9. Universal and prime models

9.1. Let $\mathcal{L} = (E)$, where E is a binary relation symbol.

(a) Define a theory T whose models are exactly the $\mathcal{L}$-structures in which E is an equivalence relation with exactly one equivalence class of size n, for each natural number $n \geq 1$.

(b) How many countable models does T have (up to isomorphism)?

(c) How many models of cardinality $\aleph_1$ does T have (up to isomorphism)?

(d) Let $\mathcal{M}_\omega$ be the model of T that is countable and has infinitely many infinite equivalence classes. Show that $\mathcal{M}_\omega$ is $\aleph_1$-universal, i.e. for every other countable model $\mathcal{N} \models T$, $\mathcal{N} \hookrightarrow_e \mathcal{M}_\omega$.

Hint: Use the proof of upward Löwenheim–Skolem to build an elementary extension of $\mathcal{N}$ with the additional requirement of having infinitely many infinite equivalence classes. Then, wake up, and realize that there is only one (up to isomorphism) such model.

(e) Conclude that T is complete.

(f) Let $\mathcal{M}_0$ be the smallest model of T, i.e. the unique model with no infinite classes; call it the *standard model*. It embeds into any other model $\mathcal{M}$ by sending the n-sized class to the n-sized class for each $n \in \mathbb{N}$; there are infinitely many such embeddings (permuting the elements of each n-sized class) and we call all of them *standard*. Show that any standard embedding is elementary, thus, $\mathcal{M}_0$ is a prime model of T.

[4] *Equinumerous* means having the same cardinality.

HINT: Use the very definition of elementarity and the completeness of T, exploiting the fact that T uniquely determines the elements of the standard part.

(g) Show that for any nonstandard model $\mathcal{M} \models T$, there are infinitely many nonstandard embeddings $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ and none of them is elementary. (This is an example that not every embedding of the prime model is elementary.) Conclude that T is not model complete.

9.2. Let $2^{<\mathbb{N}}$ denote the set of all finite binary sequences (including the empty sequence $\varnothing$) and for each $s \in 2^{<\mathbb{N}}$ and $i \in \{0,1\}$, let $s^\frown i$ be the extension of s obtained by appending the symbol i to the end of s.

Let $\mathcal{L} = (P_s)_{s\in 2^{<\mathbb{N}}}$, where each P_s is a unary predicate. Let the theory TREE comprise of the following axioms and axiom schemas:

(i) $\forall x P_\varnothing(x)$

(ii) $\exists x P_s(x)$, for each $s \in 2^{<\mathbb{N}}$

(iii) $\forall\big(P_{s^\frown 0}(x) \vee P_{s^\frown 1}(x)\big) \leftrightarrow P_s(x)$, for each $s \in 2^{<\mathbb{N}}$

(iv) $\forall x \neg\big(P_{s^\frown 0}(x) \wedge P_{s^\frown 1}(x)\big)$, for each $s \in 2^{<\mathbb{N}}$.

Below, we view every $\mathcal{L}$-structure $\mathcal{M}$ as a topological space with the topology generated by the q.f. 0-definable sets.

Prove the following:

(a) For every $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models$ TREE if and only if there is a continuous, open, topologically-injective[5] map $i_\mathcal{M} : \mathcal{M} \to 2^\mathbb{N}$ whose image is dense in $2^\mathbb{N}$.

(b) For any cardinal $\kappa \geq 1$, show that a model $\mathcal{M} \models$ TREE is κ-saturated if and only if for every $\sigma \in 2^\mathbb{N}$, $|i_\mathcal{M}^{-1}(\sigma)| \geq \kappa$.

(c) TREE admits q.e.[6] Conclude that it is complete.

HINT: One can use Blum's criterion here. If you are using any other criterion, make sure the models you deal with are sufficiently saturated (pass to elementary extensions if needed).

(d) Explicitly describe the one-dimensional type space $S_x(\text{TREE})$; what familiar topological space is it homeomorphic to? Conclude that TREE is not small, has no isolated types and no prime model.

(e) For any model $\mathcal{M} \models$ TREE and $A \subseteq M$, list all possible 1-types of $\mathcal{M}$ over A. Conclude, using q.e., that the converse of (b) holds.

(f) Conclude that a model of TREE is $\aleph_0$-saturated if and only it it is $\aleph_1$-universal.

10. ω-STABILITY AND TOTAL TRANSCENDENCE

10.1. Let $\mathcal{L}$ and T be as in Problem **9.1.**.

[5]For a topological space X and a set Y, call a map $f : X \to Y$ *topologically-injective* if the preimage $f^{-1}(y)$ of every point $y \in Y$ is not *separated by open sets*, i.e. for every open $U \subseteq X$, either $f^{-1}(y) \subseteq U$ or $f^{-1}(y) \cap U = \varnothing$.

[6]Recall that the 0-ary relation symbols $\top$ and $\bot$ for *truth* and *falsehood* are always assumed to be included in our first-order language.

(a) (Thanks to Elliot Kaplan) Add to the language $\mathcal{L}$ a unary predicate P_n for each $n \in \mathbb{N}$ and denote the new language by $\mathcal{L}'$. Let T' be the $\mathcal{L}'$-theory consisting of T together with the $\mathcal{L}'$-sentences φ_n, for all $n \in \mathbb{N}$, where φ_n says that for every x, $P_n(x)$ holds if and only if the E-class of x has exactly n elements. Show that T' admits quantifier elimination.

(b) Using the previous part, describe $S_x(T)$, as well as $S_x(\mathcal{M}/A)$ for any model $\mathcal{M} \models T$ and countable subset $A \subseteq M$. Conclude that T is ω-stable, and thus an example of an ω-stable theory that is not κ-categorical for any infinite cardinal κ.

10.2. Let $\mathcal{L}$ be a (not necessarily countable) language. For an $\mathcal{L}$-theory T and a sublanguage $\mathcal{L}_0$, let $T\restriction_{\mathcal{L}_0}$ denote its $\mathcal{L}_0$-*reduct*, i.e. $T\restriction_{\mathcal{L}_0}$ is obtained from T by removing all sentences from T that are not $\mathcal{L}_0$-sentences. Prove that a consistent and complete $\mathcal{L}$-theory T is totally transcendental if and only if $T\restriction_{\mathcal{L}_0}$ is ω-stable for all countable $\mathcal{L}_0 \subseteq \mathcal{L}$.

10.3. Let κ be an infinite cardinal and $\mathcal{L}$ be a language of size possibly larger than κ. Let T be a κ-stable $\mathcal{L}$-theory and let $\mathcal{M} \models T$ with $A \subseteq M$ of cardinality κ. Show that $\mathcal{M}$ has an elementary substructure of cardinality κ that contains A.

Remark: When $|\mathcal{L}| \leq \kappa$, this statement follows by the downward Löwenheim–Skolem theorem. However, for $|\mathcal{L}| > \kappa$, one needs an additional assumption of κ-stability.

10.4. Let T be an $\mathcal{L}$-theory.

(a) Let $\rho : S^{\mathcal{L}}_{(\vec{x},\vec{y})}(T) \to S^{\mathcal{L}}_{\vec{x}}(T)$ be the natural restriction/projection map defined by

$$p \mapsto \{\varphi : \varphi \in p \text{ and } \varphi(\vec{x}) \text{ makes sense}\},$$

where "$\varphi(\vec{x})$ makes sense" means that all free variables of φ are among $\vec{x}$. Prove that ρ is surjective, continuous, and open.

(b) Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $B \subseteq M$, $\vec{x}$ a vector of variables, and $\vec{a} \in M^{|\vec{x}|}$. Letting $\rho : S^{\mathcal{L}}_{(\vec{x},\vec{y})}(\mathcal{M}/B) \to S^{\mathcal{L}}_{\vec{x}}(\mathcal{M}/B)$ be as in (a), show that the preimage of $\mathrm{tp}_{\mathcal{M}}(\vec{a}/B)$ under this map is canonically homeomorphic to $S_{\vec{y}}(\mathcal{M}/\vec{a}B)$.

(c) Using (b), prove that for an infinite cardinal κ, T is κ-stable if and only if it is κ-stable for 1-types, i.e. for every $\mathcal{M} \models T$ and $A \subseteq \kappa$,

$$|A| \leq \kappa \Rightarrow |S_1(\mathcal{M}/A)| \leq \kappa.$$

10.5. Let T be an $\mathcal{L}$-theory and κ an infinite cardinal. Prove that if T is κ-stable, then, for all regular $\lambda \leq \kappa$, there is a λ-saturated model of T of cardinality κ.

10.6. Show that any theory with a definable infinite linear ordering (e.g. DLO) are not κ-stable for any infinite cardinal κ. Conclude that DLO, $\mathrm{Th}(\mathbb{N}; +, \cdot, 0, 1)$ and $\mathrm{Th}(\mathbb{R}; +, -, \cdot, 0, 1)$ are not κ-stable.

10.7. Prove that if T is a totally transcendental $\mathcal{L}$-theory, then for any model $\mathcal{M} \models T$ and any subset $A \subseteq M$, the space $S_1(\mathcal{M}/A)$ doesn't contain a nonempty clopen perfect subset. Conclude that the isolated types are dense in $S_1(\mathcal{M}/A)$.

11. Indiscernibles

11.1. A sequence of elements in $(\mathbb{Q};<)$ is indiscernible if and only if it is either constant, strictly increasing, or strictly decreasing.

Definition 11.2. A *Skolemization* of an $\mathcal{L}$-theory T is a theory T_S in an extended language $\mathcal{L}_S \supseteq \mathcal{L}$ that

(i) admits q.e.,
(ii) is equivalent to a universal[7] $\mathcal{L}_S$-theory,
(iii) every model $\mathcal{M}$ of T expands to a model of T_S,
(iv) $|\mathcal{L}_S| \leq \max\{|\mathcal{L}|, \aleph_0\}$.

11.3. Prove that every $\mathcal{L}$-theory T admits a Skolemization.

Hint: Extend the language in countably-many iterations, by adding Skolem functions each time, as it is done in the proof of Downward Löwenheim–Skolem theorem, which can be found in my logic notes online (Theorem 1.44). One needs to

- add to the language a k-ary function symbol $f_{\psi,k}$ for every $k \geq 0$ and every q.f. formula $\psi(\vec{x}, y)$ with $|\vec{x}| = k$,
- add
$$\forall \vec{x}\Big(\exists y \psi\big(\vec{x}, y\big) \to \psi\big(\vec{x}, f_{\psi,k}(\vec{x})\big)\Big)$$
to the theory.

The latter may not look like a universal sentence, but it becomes one once $\to$ is converted to $\vee$.

12. Prime extensions

Definition 12.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$.

- Call $\mathcal{M}$ a *prime extension over* A if, for every $\mathcal{L}$-structure $\mathcal{N}$, every elementary embedding $A \hookrightarrow_e \mathcal{N}$ (i.e. every partial elementary map $\mathcal{M} \rightharpoonup_e \mathcal{N}$ with domain A) extends to an elementary embedding $\mathcal{M} \hookrightarrow_e \mathcal{N}$.
- Call $B \subseteq M$ *constructible over* A if B admits an ordinal enumeration (i.e. a well-ordering) $B = (b_\alpha)_{\alpha<\lambda}$, for some ordinal λ, such that each b_α is atomic[8] over $A \cup B_\alpha$, where $B_\alpha := \{b_\gamma\}_{\gamma<\alpha}$.

12.2. Prove that if an $\mathcal{L}$-structure $\mathcal{M}$ is constructible over $A \subseteq M$, then $\mathcal{M}$ is prime over A.

12.3. Prove that if an $\mathcal{L}$-theory T is totally transcendental, then in any model $\mathcal{M} \models T$, every subset $A \subseteq M$ has a constructible prime extension $\mathcal{M}_0 \preceq \mathcal{M}$.

12.4. (Transitivity of being atomic) Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

(a) For any $a, b \in M$, $\mathrm{tp}(a, b)$ is isolated if and only if $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b)$ are isolated.

(b) Conclude that constructible extensions are atomic; more precisely, if $B \subseteq M$ is constructible over $A \subseteq M$, then any $b \in B$ is atomic[8] over A.

[7] A theory is called *universal* if it consists of only universal[3] sentences.
[8] For a structure $\mathcal{M}$, an element $b \in M$ is said to be *atomic over a set* $P \subseteq M$ if $\mathrm{tp}(b/P)$ is isolated in $S_1(\mathcal{M}/P)$.