CLASSIFICATION IN ERGODIC THEORY

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One of the main objects of study in ergodic theory is a probability-measure-preserving (p-m-p) dynamical system of the form (X, μ, T), where (X, μ) is a standard probability space and T a p-m-p automorphism of (X, μ), or more generally, (X, μ, Γ, α), where α is a p-m-p action of a countable group Γ on (X, μ). Naturally, we would like to classify these dynamical systems up to suitable notions of equivalence, such as isomorphism (conjugacy), unitary (spectral) equivalence, and orbit equivalence[1], by attaching invariants to these systems, such as entropy[2], spectral measures, cost. Here are some major positive results in this direction:

(1) [Ornstien 1970] Two Bernoulli shifts (X, μ, T) and (Y, ν, S) are isomorphic if and only if they have equal entropy. [Halmos–von Neumann 1942] Two dynamical systems (X, μ, T) and (Y, ν, S) with discrete spectrum are isomorphic if and only if they are unitarily equivalent if and only if their sets of eigenvalues are equal.

(2) [Dye 1963; Ornstein–Weiss 1980] Any two probability p-m-p actions of (maybe different) amenable groups are orbit equivalent.

Note that (1) classifies only special kinds of automorphisms up to isomorphism or unitary equivalence, leaving the general classification problem widely open. What if such classification was *impossible*? How would we *prove* this? This is where *descriptive set theory* enters the picture, providing a suitable framework and tools for proving non-classification results for equivalence relations.

The point of view taken here is *global*: we look at all p-m-p systems at once, i.e. we study the group $\mathrm{Aut}(X, \mu)$ of all p-m-p automorphisms, as well as the space $\mathrm{Act}(\Gamma, X, \mu)$ of all p-m-p actions of Γ on (X, μ). Here are some striking victories of this new theory:

(1′) [Hjorth 2001; Foreman–Weiss 2004] Neither isomorphism, nor unitary equivalence, admits any "reasonable" classification even if we restrict to weakly mixing automorphisms.

(2′) [Epstein–Ioana–Kechris–Tsankov 2008] If Γ is a non-amenable countable group, then it admits continuum-many non-orbit-equivalent free p-m-p actions on (X, μ). Moreover, orbit equivalence on $\mathrm{Act}(\Gamma, X, \mu)$ does not admit any "reasonable" classification.

The goal of these notes is to cover or sketch most of what is advertised above, as well as the basics of the theory of costs developed by Gaboriau and the rigidity phenomenon. Our main references are [Kec10] and [KM04].

[1] For an action $\alpha : \Gamma \curvearrowright (X, \mu)$, let E_α be the *induced orbit equivalence relation* on X. Actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Delta \curvearrowright (Y, \nu)$ of countable groups Γ, Δ are called *orbit equivalent* if E_α and E_β are measure-isomorphic, i.e. there is a measure-isomorphism $T : (X, \mu) \xrightarrow{\sim} (Y, \nu)$ such that $x E_\alpha y \Leftrightarrow T(x) E_\beta T(y)$, for μ-a.e. $x, y \in X$.

[2] The entropy of a system (X, μ, T) is a real number designed to measure its chaoticity.

Contents

Part 1. The group of measure-preserving automorphisms

1. Topologies and representations of $\mathrm{Aut}(\mu)$

Throughout this section, let (X, μ) be a standard Borel space equipped with a non-atomic Borel probability measure μ. By Theorem 7.7 any such space is isomorphic to $([0,1], \lambda)$.

1.A. **Measure-preserving automorphisms.**

For measurable maps $T, T' : (X, \mu) \to (X, \mu)$, we write $T =_\mu T'$ if they differ only on a μ-null set. We denote by $\mathrm{Aut}(X, \mu)$ the set of all bi-measurable, measure preserving automorphisms of X up to $=_\mu$, i.e. we identify two such automorphisms T, T' if $T =_\mu T'$. We denote the $=_\mu$-equivalence class of T by $[T]_\mu$ or simply by $[T]$; below, we abuse the notation even further and just denote it by T as we normally do in analysis for functions in L^p spaces.

Lemma 1.1 (Replacing measurable with Borel)**.** *For every $[T] \in \mathrm{Aut}(X, \mu)$ there is a Borel automorphism T' of X with $T =_\mu T'$.*

Proof. Let $\{U_n\}_n$ be a countable basis for X. So, for every n the set $T^{-1}(U_n)$ is measurable. Let B_n be a Borel set with $B_n =_\mu T^{-1}(U_n)$ and let $X' = \bigcup_n B_n$. Notice that X' is Borel and that $X =_\mu X'$. □

Alternative proof. Let $\Phi_T : \mathrm{MALG}_\mu \to \mathrm{MALG}_\mu$ be given by $\Phi_T([A]) = [T^{-1}(A)]$ and using Corollary 7.14, let T' be a Borel automorphism that induces Φ_T. □

1.2. *As elements of* $\mathrm{Aut}(\mathrm{MALG}_\mu, \mu)$. Recall from Subsection 7.F that we can view each $[T] \in \mathrm{Aut}(X, \mu)$ as an element of $\mathrm{Aut}(\mathrm{MALG}_\mu)$, the group of all σ-automorphisms of MALG_μ, by $[T] \mapsto \Phi_T$, where $\Phi_T([A]) = [T^{-1}(A)]$ for $A \in \mathrm{MEAS}_\mu$. Moreover, Corollary 7.14 allows us to identify $\mathrm{Aut}(X, \mu)$ with the subgroup $\mathrm{Aut}(\mathrm{MALG}_\mu, \mu)$ of $\mathrm{Aut}(\mathrm{MALG}_\mu)$ of μ-preserving σ-automorphisms, i.e. $\mu(\Phi([A])) = \mu([A])$. Thus, we simply write $\mathrm{Aut}(\mu)$ for $\mathrm{Aut}(X, \mu)$.

1.B. **The weak topology on $\mathrm{Aut}(\mu)$.**

In what follows, we will study the pointwise convergence topology on $\mathrm{Aut}(\mu)$, also known as the *weak topology* on $\mathrm{Aut}(\mu)$. Recall the metric d_μ on MALG_μ given by $d_\mu([A], [B]) = \mu(A \triangle B)$. We equip $\mathrm{Aut}(\mathrm{MALG}_\mu, d_\mu)$ with the *pointwise convergence topology*, where the points are the elements of MALG_μ, i.e. it is the smallest topology that makes all maps $T \to T(A)$ continuous for every $A \in \mathrm{MALG}_\mu$.

1.3. *Basis and convergence.* A basis of this topology consists of finite intersections of sets of the form

$$B_A(S, r) = \{T \in \mathrm{Aut}(\mathrm{MALG}_\mu, d_\mu) : d_\mu(T(A), S(A)) < r\},$$

where $A \in \mathrm{MALG}_\mu$, $S \in \mathrm{Aut}(\mathrm{MALG}_\mu, d_\mu)$ and $r \in [0,1]$. In this topology, we have that $T_n \to T$ (we also write $T_n \to_w T$) if and only if $T_n(A) \to T(A)$ for every $A \in \mathrm{MALG}_\mu$. Moreover, it suffices to check that $T_n(A) \to T(A)$ for a dense subset of A's in MALG_μ.

1.4. *As a topological group.* One important fact about the weak topology is that it makes $\mathrm{Aut}(\mathrm{MALG}_\mu, d_\mu)$ a topological group, i.e. composition $\mathrm{Aut}(\mu) \times \mathrm{Aut}(\mu) \to \mathrm{Aut}(\mu)$ and inversion $\mathrm{Aut}(\mu) \to \mathrm{Aut}(\mu)$ maps are continuous in this topology; indeed, let $T_n \to_w T$ and $S_n \to_w S$. Fix $A \in \mathrm{MALG}_\mu$ and pick $\varepsilon > 0$. Chose $n \in \mathbb{N}$ large enough so that both $d_\mu\big(T_n(A), T(A)\big) < \varepsilon/2$ and $d_\mu\big(S_n \circ T(A), S \circ T(A)\big) < \varepsilon/2$. Using the triangle inequality, we get that $d_\mu\big(S_n \circ T_n(A), S \circ T(A)\big) < \varepsilon$. Moreover, since $T_n \to_w T$, we have that $T_n \circ T^{-1}(A) \to T \circ T^{-1}(A) = A$. So,

$$d_\mu\big(T_n^{-1}(A), T^{-1}(A)\big) = d_\mu\big(T_n \circ T_n^{-1}(A), T_n \circ T^{-1}(A)\big) = d_\mu\big(A, T_n \circ T^{-1}(A)\big) \to 0.$$

1.5. *Compatible metrics.* Let $\{A_n\}_n$ be a dense subset of $(\mathrm{MALG}_\mu, d_\mu)$. The weak topology is induced by the following metric:

$$\delta_w(T,S) := \sum_n 2^{-n} d_\mu\big(T(A_n), S(A_n)\big).$$

Note that δ_w is a left invariant metric, i.e. $\delta_w(S \circ T_1, S \circ T_2) = \delta_w(T_1, T_2)$.

However, δ_w is not a complete metric: Given a δ_w-Cauchy sequence $(T_n)_n$, one can define its limit by $T(A) := \lim_n T_n(A)$. This would indeed define a one-to-one measurable endomorphism of (MALG_μ, μ), but it may not be bijective. To avoid this, we would like $(T_n^{-1})_n$ to also δ_w-Cauchy. Hence, we switch to an equivalent metric, which would have the property that if $(T_n)_n$ is Cauchy, so is $(T_n^{-1})_n$.

Thus, we define a metric $\overline{\delta_w}$ on $\mathrm{Aut}(\mathrm{MALG}_\mu, d_\mu)$ as follows:

$$\overline{\delta_w}(T,S) := \delta_w(T,S) + \delta_w(T^{-1}, S^{-1}).$$

To see that this metric is complete let $(T_n)_n$ be a $\overline{\delta_w}$-Cauchy sequence and notice that $\{T_n^{-1}\}$ is also $\overline{\delta_w}$-Cauchy. Define $T, S : \mathrm{MALG}_\mu \to \mathrm{MALG}_\mu$ by $T(A) = \lim T_n(A)$ and $S(A) = \lim T_n^{-1}(A)$. As an exercise, check that $T_n \to_w T$, $T_n^{-1} \to_w S$, $S \circ T = T \circ S = \mathrm{id}_{\mathrm{MALG}_\mu}$ and therefore both T, S are indeed elements of $\mathrm{Aut}(\mu)$.

1.6. *As a closed subgroup of* $\mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$. For sets $A, B \in \mathrm{MALG}_\mu$, we often write $A \perp B$ to mean that A and B are disjoint (mod NULL_μ).

Lemma 1.7. *The topological group* $\mathrm{Aut}(\mu)$ *is a closed subgroup of* $\mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$. *In fact, it is the group of all* $T \in \mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$ *with* $T(\varnothing) = \varnothing$.

Proof. We use the identification $\mathrm{Aut}(\mu) = \mathrm{Aut}(\mathrm{MALG}_\mu, \mu)$. For a μ-preserving σ-automorphism $T \in \mathrm{Aut}(\mathrm{MALG}_\mu, \mu)$, we check that it preserves the metric d_μ; indeed, for $A, B \in \mathrm{MALG}_\mu$,

$$d_\mu(T(A), T(B)) = \mu\big(T(A) \triangle T(B)\big) = \mu\big(T(A \triangle B)\big) = \mu\big(A \triangle B\big) = d_\mu(A,B).$$

To check the converse, fix $T \in \mathrm{Iso}(\mathrm{MALG}_\mu, d_\mu)$ with $T(\varnothing) = \varnothing$. We need to check that T is a σ-automorphism of MALG_μ and it preserves μ. The latter is immediate: for $A \in \mathrm{MALG}_\mu$,

$$\mu(T(A)) = \mu(T(A) \triangle \varnothing) = d_\mu(T(A), \varnothing) = d_\mu(T(A), T(\varnothing)) = d_\mu(A, \varnothing) = \mu(A).$$

As sets $A, B \in \mathrm{MALG}_\mu$ are disjoint if and only $d_\mu(A,B) = \mu(A) + \mu(B)$, T maps disjoint sets to disjoint sets. Applying this to $A, -A$ and using that T preserves μ, we see that T preserves complements, and thus also the subset relation $\subseteq$ because $A \subseteq B \Leftrightarrow A \perp (-B)$. To see that T preserves (finite) joint $\vee$, notice that since it preserves $\subseteq$, we have that $T(A), T(B) \subseteq T(A \vee B)$ and therefore

$$T(A) \vee T(B) \subseteq T(A \vee B). \tag{$*$}$$

Applying T^{-1} to both sides of this, we get

$$T^{-1}\big(T(A) \vee T(B)\big) \subseteq A \vee B. \tag{$**$}$$

Applying $(*)$ to $A' = T(A)$, $B' = T(B)$ and $T' = T^{-1}$ and using $(**)$, we get

$$T(A \vee B) = T^{-1}(T(A)) \vee T^{-1}(T(B)) \subseteq T^{-1}\big(T(A) \vee T(B)\big) \subseteq A \vee B.$$

So $A \vee B = T^{-1}\big(T(A) \vee T(B)\big)$, and therefore $T(A \vee B) = T(A) \vee T(B)$.

Finally, because T is d_μ-continuous and $\bigvee_{i<n} A_i \to_{d_\mu} \bigvee_{i\in\mathbb{N}} A_i$ as $n \to \infty$, we get that T also preserves countable joint. $\square$

This, together with Proposition 4.6, gives:

Corollary 1.8. *The group* $\mathrm{Aut}(\mu)$ *with the weak topology is a Polish group.*

1.C. **The unitary group of $\mathcal{H} = L^2(X,\mu)$.** Consider the Hilbert space $\mathcal{H} = \mathrm{L}^2(X,\mu)$. Let $B(\mathcal{H})$ be the space of all bounded linear operators $L : \mathcal{H} \to \mathcal{H}$ and let $U(\mathcal{H}) \subseteq B(\mathcal{H})$ be the the group of all unitary operators on $\mathcal{H}$, i.e., all $U \in B(\mathcal{H})$ that preserve the norm $\|\cdot\|_2$ (or equivalently, using the parallelogram equality, the inner product). We now recall various topologies on $U(\mathcal{H})$.

1.9. *The operator topology.* Consider the topology τ_{op} on $B(\mathcal{H})$ induced by the operator norm

$$\|L\|_{op} := \sup_{\|h\|_2=1} \|L(h)\|_2.$$

Restricting this topology to $U(\mathcal{H})$ we get the operator topology on $U(\mathcal{H})$. This topology admits a metric, namely, the one induced by the norm on $B(\mathcal{H})$, and it is easy to see that this metric is a complete metric on $U(\mathcal{H})$. However, $(U(\mathcal{H}), \tau_{op})$ is not separable. To see that let $(e_n)_n$ be an orthonormal basis and notice that distinct permutations of this basis give distinct unitaries which are pairwise $\sqrt{2}$ apart in the norm metric.

1.10. *The strong and weak topologies.* The strong topology on $B(\mathcal{H})$ is the one generated by the family of maps $L \mapsto L(h)$, where $h \in \mathcal{H}$, and $\mathcal{H}$ is endowed with the norm topology. Therefore,

$$L_n \to_s L \iff \forall h \in \mathcal{H} \quad L_n(h) \to_{\mathcal{H}} L(h).$$

The weak topology on $B(\mathcal{H})$ is the one generated by the family of maps $L \mapsto L(h)$, where $h \in \mathcal{H}$, and $\mathcal{H}$ is endowed with the weak topology. In other words, The weak topology on $B(\mathcal{H})$ is the one generated by the family of maps $L \mapsto \langle L(h), g\rangle$, where $h, g \in \mathcal{H}$. Therefore,

$$L_n \to_w L \iff \forall h, g \in \mathcal{H} \quad \langle L_n(h), g\rangle \to \langle L(h), g\rangle.$$

It is clear that the strong topology is finer (not coarser) than the weak topology. However, we have:

Proposition 1.11. *On* $U(\mathcal{H})$*, the weak and strong topologies coincide.*

Proof. We need to prove that weak convergence implies strong convergence. For that assume that $U_n \to_w U$ and fix $h \in \mathcal{H}$. We have that

$$\begin{aligned} \|U(h) - U_n(h)\|^2 &= \langle U(h) - U_n(h), U(h) - U_n(h)\rangle \\ &= \|U(h)\|^2 + \|U_n(h)\|^2 - 2\langle U_n(h), U(h)\rangle \\ [U_n, U \text{ preserve } \|\cdot\|] \quad &= \|h\|^2 + \|h\|^2 - 2\langle U_n(h), U(h)\rangle \\ [\text{weak convergence}] \quad &\to \|h\|^2 + \|h\|^2 - 2\langle U(h), U(h)\rangle \\ [U \text{ preserves } \langle\cdot,\cdot\rangle] \quad &= \|h\|^2 + \|h\|^2 - 2\langle h, h\rangle = 0. \end{aligned}$$

□

1.D. **Koopman representation.** For every $T \in \mathrm{Aut}(\mu)$ we associate $U_T \in U(\mathcal{H})$ by $U_T f = f \circ T^{-1}$, for $f \in \mathcal{H}$. Notice that indeed each U_T as above is unitary since

$$\begin{aligned} \|U_T f\|^2 &= \int_X \left(f \circ T^{-1}(x)\right)^2 d\mu(x) \\ [T \text{ preserves } \mu, \text{ change of variable } x \mapsto T(x)] \quad &= \int_X |f(x)|^2 d\mu(x) = \|f\|^2, \end{aligned}$$

and U_T is invertible with $U_T^{-1} = U_{T^{-1}}$.

Observe now that the map $T \to U_T$ is a topological group embedding from $\mathrm{Aut}(\mu)$ with the weak (= pointwise convergence) topology to $U(\mathcal{H})$ with the strong (= pointwise convergence = weak) topology. This map is called the *Koopman representation*.

Proposition 1.12. *The image of* $\mathrm{Aut}(\mu)$ *under the Koopman representation is exactly the set of all multiplicative unitary operators, i.e. all* $U \in U(\mathcal{H})$ *satisfying* $U(f\cdot g) = U(f)\cdot U(g)$ *for every* $f, g \in \mathcal{H}$.

Proof. Clearly, for each $T \in \mathrm{Aut}(\mu)$, we have that

$$U_T(f\cdot g) = (f\cdot g)\circ T^{-1} = (f \circ T^{-1})\cdot(g\circ T^{-1}) = U_T(f)\cdot U_T(g).$$

Now let $U \in U(\mathcal{H})$ be multiplicative. Then for every $A \in \mathrm{MALG}_\mu$,

$$U(\mathbb{1}_A) = U(\mathbb{1}_A \cdot \mathbb{1}_A) = U(\mathbb{1}_A)\cdot U(\mathbb{1}_A),$$

so $U(\mathbb{1}_A) = \mathbb{1}_B$ for some $B \in \mathrm{MALG}_\mu$. Moreover, since U preserves the norm, we have that $\mu(A) = \mu(B)$. Now define T_U by $T_U(A) \coloneqq B$ where $U(\mathbb{1}_A) = \mathbb{1}_B$. Notice that T_U respects intersections since $U(\mathbb{1}_{A\cap B}) = U(\mathbb{1}_A \cdot \mathbb{1}_B) = U(\mathbb{1}_A)\cdot U(\mathbb{1}_B) = \mathbb{1}_{A'}\cdot \mathbb{1}_{B'} = \mathbb{1}_{A'\cap B'}$. Since T is measure-preserving and respects intersections, it also respects complements. Finally, the continuity of U implies that T also respects countable intersections. □

We also have the following alternative characterization, whose proof is left as an exercise.

Proposition 1.13. *The image of* $\mathrm{Aut}(\mu)$ *under the Koopman representation is exactly the set of all positive operators (i.e.* $f \geq 0 \Rightarrow U(f) \geq 0$*) that fix* $\mathbb{1}_X$.

Now let $\mathbb{C}\cdot\mathbb{1}_X$ be the set of all constant functions in $\mathcal{H}$ and let

$$L_0^2(X,\mu) = \left(\mathbb{C}\cdot \mathbb{1}_X\right)^\perp = \left\{ f \in L^2(X,\mu) : \int f d\mu = 0 \right\}.$$

Putting $U_T^0 \coloneqq U_T \restriction_{\mathcal{L}_0^2(X,\mu)}$, we see that $T \mapsto U_T^0$ is an embedding of $\mathrm{Aut}(\mu)$ into $U\big(L_0^2(X,\mu)\big)$. We record here the following fact and refer to [Gla03, 5.14] for a proof.

Theorem 1.14. *The representation* $\mathrm{Aut}(\mu) \hookrightarrow U\big(L^2_0(X,\mu)\big)$ *is irreducible, i.e. the action* $\mathrm{Aut}(\mu) \curvearrowright L^2_0(X,\mu)$ *does not have non-trivial invariant subspaces.*

1.E. **The uniform topology on** $\mathrm{Aut}(\mu)$**.** For $S,T \in \mathrm{Aut}(\mu)$ let

$$\delta'_u(S,T) = \sup\left\{d_\mu\big(T(A),S(A)\big) : A \in \mathrm{MALG}_\mu\right\}.$$

Notice that δ'_u is a 2-sided invariant metric on $\mathrm{Aut}(\mu)$. Hence,

$$\delta'_u(S,T) = \delta'_u(T^{-1}S, \mathrm{id}_{\mathrm{MALG}_\mu}) = \delta'_u(T^{-1},S^{-1}).$$

As a consequence, δ'_u is a complete metric since $(T_n)_n$ being δ'_u-Cauchy implies that $(T_n^{-1})_n$ is also δ'_u-Cauchy, so letting T,S be the obvious guesses for limits of $(T_n)_n$ and $(T_n^{-1})_n$ (as done in the proof of the completeness of $\overline{\delta}_w$), we see that indeed $T^{-1} = S$, so $T \in \mathrm{Aut}(\mu)$. The topology τ_u on $\mathrm{Aut}(\mu)$ induced by δ'_u is called the *uniform topology* of $\mathrm{Aut}(\mu)$.

Sometimes, it is more convenient to work with the following metric, which, as shown below, is equivalent to δ'_u. For $S,T \in \mathrm{Aut}(\mu)$, let $D(S,T) = \{x \in X : S(x) \neq T(x)\}$. Let also

$$\delta_u(S,T) = \mu\big(D(S,T)\big).$$

We will now work towards establishing the following relation between these two "uniform" metrics:

$$\frac{2}{3}\delta_u \leq \delta'_u \leq \delta_u.$$

Lemma 1.15. $\delta'_u \leq \delta_u$.

Proof. Let $A \in \mathrm{MALG}_\mu$, $B_S = S(A) \smallsetminus T(A)$ and $B_T = T(A) \smallsetminus S(A)$. Notice that if $C = T^{-1}(B_S \cup B_T)$ then $C \subset D(S,T)$ and therefore

$$d_\mu(S(A),T(A)) = \mu(B_S \cup B_T) = \mu(C) \leq \mu\big(D(S,T)\big) = \delta_u(S,T).$$

□

Lemma 1.16. *For any distinct* $S,T \in \mathrm{Aut}(\mu)$*, there is a* μ*-positive* $A \in \mathrm{MALG}_\mu$ *with* $S(A) \perp T(A)$.

Proof. Take $0 < \varepsilon < \delta_u(S,T)$ and a subset $X' \subseteq X$ with $\mu(X') > 1 - \varepsilon$ such that $S\restriction_{X'}$ and $T\restriction_{X'}$ are continuous. Notice that $X' \cap D(S,T) \neq \varnothing$. Fix a countable open basis $\{U_n\}_n$ of X and note that by the continuity of S,T, for every $x \in X' \cap D(S,T)$ there is $n_x \in \mathbb{N}$ with $x \in U_{n_x}$ and $S(U_{n_x} \cap X') \perp T(U_{n_x} \cap X')$. Since the family $\{U_{n_x} : x \in X' \cap D(S,T)\}$ covers $X' \cap D(S,T)$ and the latter is μ-positive, there is some n_x such that $X' \cap U_{n_x}$ is μ-positive. It is clear now that $A \coloneqq U_{n_x} \cap X'$ is as desired. □

Lemma 1.17. *Any nonempty* $\mathcal{F} \subseteq \mathrm{MALG}_\mu$ *that is closed under countable increasing unions has a* $\subseteq$*-maximal element.*

Proof. We recursively construct an increasing sequence $A_n \in \mathcal{F}$ as follows. Take arbitrary $A_0 \in \mathcal{F}$. Given A_n, take $A_{n+1} \supseteq A_n$ with

$$\mu(A_{n+1} \smallsetminus A_n) \geq \frac{1}{2}\sup\{\mu(B \smallsetminus A_n) : B \in \mathcal{F}\}.$$

By the hypothesis on $\mathcal{F}$, $A \coloneqq \bigcup_n A_n \in \mathcal{F}$ and it is straightforward to check that A is $\subseteq$-maximal. □

Lemma 1.18. *For any distinct* $S,T \in \mathrm{Aut}(\mu)$*, there is* $A \in \mathrm{MALG}_\mu(X,\mu)$ *with* $S(A) \perp T(A)$ *and* $\mu(A) \geq \frac{1}{3}\delta_u(S,T)$.

Proof. Let $\mathcal{F}$ be the set of all $A \in \mathrm{MALG}_\mu(X,\mu)$ with $S(A) \perp T(A)$ and note that $\varnothing \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable increasing unions. Let A be an $\subseteq$-maximal element of $\mathcal{F}$, which exists by Lemma 1.17. We claim that $\mu(A) \geq \frac{1}{3}\delta_u(S,T)$. Suppose not. Then, for the set

$$D' \coloneqq D(S,T) \setminus \Big(A \cup \big(S^{-1} \circ T\big)(A) \cup \big(T^{-1} \circ S\big)(A)\Big)$$

we have $\mu(D') \geq \mu\big(D(S,T)\big) - 3\mu(A) > 0$. By Lemma 1.16, we now get $B \subset D'$ with $\mu(B) > 0$ and $S(B) \perp T(B)$. Note that the sets $S(B)$, $T(B)$, $T(A)$, $S(A)$ are pairwise disjoint, and therefore $S(A \cup B) \perp T(A \cup B)$, contradicting the $\subseteq$-maximality of A. $\square$

Remark 1.19. The $\frac{1}{3}$ in the above lemma is sharp: let $S = \mathrm{id}_{[0,1]}$ and let $T : [0,1] \to [0,1]$ be given by $x \mapsto x + 1/3 \,(\mathrm{mod}\, 1)$. Then $\delta_u(S,T) = 1$ and we leave it as an exercise to check that for any $A \subseteq [0,1]$ with $\lambda(A) > 1/3$, A and $T(A)$ have λ-positive intersection.

Lemmas 1.15 and 1.18 together imply:

Corollary 1.20. *The metrics δ'_u and δ_u are equivalent. In fact,*

$$\frac{2}{3}\delta_u \leq \delta'_u \leq \delta_u.$$

Proof. We only need to show that first inequality, so let $S,T \in \mathrm{Aut}(\mu)$ and let $A \in \mathrm{MALG}_\mu$ be a set given by Lemma 1.18, i.e. $S(A) \perp T(A)$ and $\mu(A) \geq \delta_u(S,T)$. Then

$$\delta'_u(S,T) \geq d_\mu(S(A),T(A)) = \mu(S(A)) + \mu(T(A)) = 2\mu(A) \geq \frac{2}{3}\delta_u(S,T).$$

$\square$

Corollary 1.21. *The uniform topology on* $\mathrm{Aut}(\mu)$ *is not separable.*

Proof. Let (X,μ) be $([0,1],\lambda)$ and let $\mathcal{F} = \{T_\alpha : \alpha \in [0,1)\} \subseteq \mathrm{Aut}(\mu)$ with $T_\alpha(x) = x + \alpha \,(\mathrm{mod}\, 1)$. Then $\mathcal{F}$ has cardinality continuum, yet for distinct $\alpha, \beta \in [0,1)$, $\delta_u(T_\alpha, T_\beta) = 1$. $\square$

Proposition 1.22. *The closed balls in* $\mathrm{Aut}(\mu)$ *with respect to either of the metrics δ_u and δ'_u are closed in the weak topology.*

Proof. For δ'_u, note that $T \in \overline{B}_{\delta'_u}(S,r) \iff \forall A \in \mathrm{MALG}_\mu \; d_\mu(S(A),T(A)) \leq r$. The proof for δ_u is slightly more involved and we will omit it here. $\square$

Remark 1.23. Viewing $\mathrm{Aut}(\mu)$ inside $U(\mathcal{H})$ via the Koopman representation, we could endow $\mathrm{Aut}(\mu)$ with the operator norm topology inherited from $U(\mathcal{H})$. However this topology is not useful as it is actually the discrete topology. Indeed, it is enough to show that for every $\mathrm{id}_{\mathrm{MALG}_\mu} \neq T \in \mathrm{Aut}(\mu)$, $\|U_T - I\| \geq 1$. Using Lemma 1.16, get $A \in \mathrm{MALG}_\mu(X,\mu)$ with $\mu(A) > 0$ and $T(A) \perp A$. Then, putting $f \coloneqq \frac{1}{\sqrt{\mu(a)}}\mathbb{1}_A$ so $\|f\|_2 = 1$, we compute that $\|U_T(f) - f\|_2 = 1$, so $\|U_T - I\| \geq 1$.

2. Dense and generic families in $\mathrm{Aut}(\mu)$

For $T, S \in \mathrm{Aut}(\mu)$ we have by definition that $(X,\mu,T) \simeq (X,\mu,S)$ if and only if there is a measurable isomorphism $\varphi : (X,\mu) \to (X,\mu)$ such that $\varphi \circ T = S \circ \varphi$ or else $\varphi \circ T \circ \varphi^{-1} = S$. In other words $\simeq$ in $\mathrm{Aut}(\mu)$ is just the conjugacy relation on $\mathrm{Aut}(\mu)$. Also, viewing $\mathrm{Aut}(\mu)$ as a subgroup of $U(L^2(X,\mu))$ we have a coarser equivalence relation $\simeq_U$ on $\mathrm{Aut}(\mu)$ which we call unitary equivalence. This is given by the conjugacy relation on $U(L^2(X,\mu))$, i.e. $T \simeq_U S$ if and only if there is $U \in U(L^2(X,\mu))$ such that $U \circ T \circ U^{-1} = S$. (Here we need that $\mathrm{Aut}(\mu)$ is normal subgroup of $U(\mathcal{H})$ in the Koopman representation)

In the next subsections we will develop the tools that we need in order to prove that the conjugacy relation and the unitary equivalence on $\mathrm{Aut}(\mu)$ are non-smooth. Since $\simeq$ is finer than $\simeq_U$ then by Theorem 8.26 and Corollary 11.8 this amounts in showing that:

- conjugacy classes are dense in $\mathrm{Aut}(\mu)$ (in the weak topology), and
- unitary classes are meager.

Again, without the loss of generality it suffices to prove this in the case where $X = 2^{\mathbb{N}}$ and μ is the coin flip measure, i.e. the unique regular measure on $2^{\mathbb{N}}$ for which $\mu(N_s) = 2^{-|s|}$ for every $s \in 2^{<\mathbb{N}}$.

2.A. **Dyadic permutations.** For every $n \in \mathbb{N}$ we denote by $\Sigma(2^n)$ the group of all permutations on the set 2^n of all binary sequences s of length n. For every $\pi \in \Sigma(2^n)$ and $s \in 2^n$ let $T_\pi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be defined by $s^\frown x \mapsto \pi(s)^\frown x$. All such T_π are called *dyadic permutations* of length n. We call T_π a *cyclic permutation* if π is cyclic. A *dyadic interval* of rank n is some N_s with $|s| = n$. A *dyadic set* of rank n is a union of dyadic intervals of rank n. The following proposition follows directly from the definition of the coin-flip measure.

Proposition 2.1. *The family of dyadic sets is dense in* MALG_μ.

For the proof of Theorem 2.3 we first need a lemma.

Lemma 2.2. *The set of dyadic permutations is dense in* $\mathrm{Aut}(\mu)$ *in the weak topology.*

Proof. (to be made more precise)

Recall that a basis of open sets of $\mathrm{Aut}(\mu)$ consists of the sets of the form

$$B(S, A_1, \dots, A_k, r) := \{T \in \mathrm{Aut}(\mu) \ : \ \forall\, i \le k \ \ d_\mu\left(T(A_i), S(A_i)\right) < r\}$$

By replacing $A_1, \dots, A_k$ above by the atoms of the finite Boolean algebra they generate, we can assume without the loss of generality that $A_1, \dots, A_k$ form a partition of $2^{\mathbb{N}}$. Notice now that we have two finer partitions $\{A_i \cap S^{-1}(A_j) : 1 \le i, j \le k\}$ and $\{S(A_i) \cap A_j : 1 \le i, j \le k\}$ of $2^{\mathbb{N}}$. Using 2.1 we can find dyadic sets U_{ij} and V_{ij} with $1 \le i, j \le k$ so that

1. (1) $\mu(U_{ij}) = \mu(V_{ij})$;
2. (2) U_{ij} is "d_μ-close" to $A_i \cap S^{-1}(A_j)$;
3. (3) V_{ij} is "d_μ-close" to $A_i \cap S^{-1}(A_j)$;
4. (4) $U_{ij} \cap U_{i'j'} = \varnothing$ if $i \neq i'$ or $j \neq j'$;
5. (5) $V_{ij} \cap V_{i'j'} = \varnothing$ if $i \neq i'$ or $j \neq j'$ and
6. (6) $\bigcup_{ij} U_{ij}$ and $\bigcup_{ij} V_{ij}$ are "d_μ-close" to X.

All the properties above can be fulfilled simultaneously accumulating an total error within the r-room that we have. Since we used only finitely many dyadic sets, there is a number $n \in \mathbb{N}$ so that all U_{ij}'s and V_{ij}'s as well well as $2^{\mathbb{N}} \setminus \bigcup_{ij} U_{ij}$ and $2^{\mathbb{N}} \setminus \bigcup_{ij} V_{ij}$ are all dyadic sets of rank n. Let now T_π to be any dyadic permutation with $\pi \in \Sigma(2^n)$ so that $T_\pi(U_{ij}) = V_{ij}$. $\square$

Theorem 2.3. *The set of cyclic dyadic permutations is dense in* $\mathrm{Aut}(\mu)$ *in the weak topology. In fact in every weakly open set* U *in* $\mathrm{Aut}(\mu)$ *we can find a cyclic dyadic permutation of rank* n *for all but finitely many* $n \in \mathbb{N}$.

Proof. Let U be a weakly open subset of $\mathrm{Aut}(\mu)$. By Lemma 2.2 and Proposition 2.1 we can assume that

$$U = B(S, A_1, \ldots, A_k, \varepsilon) := \{T \in \mathrm{Aut}(\mu) \ : \ \forall\, i \le k \ \ d_\mu\left(T(A_i), S(A_i)\right) < \varepsilon\}$$

where $A_1, \ldots, A_k$ forms a dyadic partition of $2^{\mathbb{N}}$ and S is a dyadic permutation of rank say m. Actually, we can assume without the loss of generality that the sets $A_1, \ldots, A_k$ are exactly the partition $\{N_s : s \in 2^m\}$ where m is the rank of S. **this should be written as a corollary of Lemma 2.2**

So, $S = T_\pi$ for some $\pi \in \Sigma(2^m)$ where in general, π is a product $\gamma_1 \cdot \ldots \cdot \gamma_k$ of disjoint dyadic cycles $\gamma_i = (s^i_1, \ldots, s^i_{l_i})$ with $s^i_{l_j} \in 2^m$ for every j. Let n_0 be such so that $n_0 > m$ and $2^{-n_0} < \frac{\varepsilon}{k}$.

We proceed as follows. Every $s \in 2^m$ appears in a unique position in the expression of π below:

$$\pi = \overbrace{(s^1_1, \ldots, s^1_{l_1})}^{\gamma_1} \cdot \overbrace{(s^2_1, \ldots, s^2_{l_2})}^{\gamma_2} \cdot \ldots \cdot \overbrace{(s^k_1, \ldots, s^k_{l_k})}^{\gamma_k}$$

Let any $n > n_0$ and set $p = 2^{n-m}$. Let also $t_1, t_2, \ldots, t_p$ be the unique enumeration of all $t \in 2^{n-m}$ that respects the lexicographic order. Notice that for every $s \in 2^m$ and every t from the list above $s^\frown t$ belongs to 2^n and moreover the collection of all $s^\frown t$ partitions 2^n. Finally notice that for every $s^\frown t$ as above we have $\mu(N_{s^\frown t}) = 2^{-n} < 2^{-n_0} < \frac{\varepsilon}{k}$. We define $\sigma \in \Sigma(2^n)$ as follows:

- for any $t \in \{t_1, \ldots, t_p\}$ and if s is not any of the terminal points in the above cycles i.e, $s \notin \{s^1_{l_1}, s^2_{l_2}, \ldots, s^k_{l_k}\}$, then set $\sigma(s^\frown t) = \pi(s)^\frown t$;
- for $t = t_i \neq t_p$ and s being some of the terminal points in the above cycles i.e, $s \in \{s^1_{l_1}, s^2_{l_2}, \ldots, s^k_{l_k}\}$, set $\sigma(s^\frown t_i) = \pi(s)^\frown t_{i+1}$;
- finally if $t = t_p$ and s is some of the terminal points in the above cycles i.e, $s = s^i_{l_i}$ for some $i \in \{1, \ldots, k\}$, then set $\sigma(s^i_{l_i}{}^\frown t_p) = s^{i+1}_1{}^\frown t_1$.

Let now $T = T_\sigma$. Then T is a a cyclic dyadic permutation of rank n which lies entirely inside U. (hmm we didn't need to take ε/k is seems...) ☐

2.B. **Aperiodic automorphisms.** Let $T \in \mathrm{Aut}(\mu)$ and $x \in X$. We say that the period of x under T is n with $n > 0$ if n is the smallest number so that $T^n(x) = x$. We say that the period of x under T is ∞ if for all $n > 1$ we have $T^n(x) \neq x$. If there is an $n > 1$ so that for μ-almost all $x \in X$ the period of x is n, then we say that T is a *periodic* automorphism of period n. We have seen that a very special class of periodic permutations on $2^{\mathbb{N}}$, namely the cyclic dyadic permutations, is dense in the $\mathrm{Aut}(\mu)$ in the weak topology. Similarly a $T \in \mathrm{Aut}(\mu)$ is called *aperiodic* if for μ-almost all $x \in X$ the period of x is ∞. We denote the set of all aperiodic automorphisms of $\mathrm{Aut}(\mu)$ by APER. Notice that

$$T \in \mathrm{APER} \iff \forall n > 0 \ \ \delta_u(T^n, \mathrm{id}_X) = 1 \iff \forall n > 0 \ \ \forall m > 0 \ \ \delta_u(T^n, \mathrm{id}_X) > 1 - \frac{1}{m},$$

where with a careful look the last set is weakly open (see also proposition 1.22). Therefore APER is uniformly closed and weakly a G_δ set.

Lemma 2.4. *The set* APER *is a dense subset of* $\mathrm{Aut}(\mu)$*.*

Proof. It is enough by 2.2 to approximate each dyadic permutation with an aperiodic one. Fix therefore a $\pi \in \Sigma(2^n)$ and let $U = B(T_\pi, A_1, \ldots, A_k, \varepsilon)$ be a weak open neighborhood of T_π. We can assume without the loss of generality that $A_1, \ldots, A_k$ are dyadic interval of some rank m and we may assume that $n \geq m$. So we may take as well $n = m$. Let now T be any aperiodic transformation and notice that the automorphism S given by

$$S(s^\frown x) := \pi(s)^\frown T(x), \quad \text{with } |s| = n,$$

is aperiodic and belongs to U. □

Corollary 2.5. *A generic automorphism of* $\mathrm{Aut}(\mu)$ *belongs to* APER.

2.C. **Uniform approximation and conjugacy.** Our aim here is to prove that for every $T \in \mathrm{APER}$, the conjugacy class $[T]_{\mathrm{conj}}$ of T is dense in $\mathrm{Aut}(\mu)$. For this, we first show that we can always uniformly approximate each $T \in \mathrm{APER}$ with periodic automorphisms. Since any two periodic automorphisms S, S' of same period n are conjugate one another, we can use this approximation to control the conjugacy class of T.

Recall that for $T \in \mathrm{Aut}(\mu)$ we denote by E_T the induced orbit equivalence relation on X.

Lemma 2.6. *Let* $S, T \in \mathrm{Aut}(\mu)$ *be two periodic automorphisms both of period* n. *Then* S, T *are conjugate one another.*

Proof. Using Proposition 12.4, we can pick Borel transversals $A_S, A_T \subseteq X$ for the finite equivalence relations E_S and E_T. Notice that $\mu(A_S) = \mu(A_T) = 1/n$. By the measure isomorphism theorem 7.7 let $Q_0 : A_S \to A_T$ be a measure isomorphism. Extend Q to X by letting $Q{\restriction}_{S^i(A_S)} = T^i \circ Q_0 \circ S^{-i}$ and notice that $T = Q \circ S \circ Q^{-1}$. □

Next we prove a technical lemma that we need for the main theorem.

Lemma 2.7 (Rokhlin Lemma)**.** *For every* $T \in \mathrm{APER}$, *for every* $\varepsilon > 0$ *and* $n \geq 1$ *there is a Borel* E_T*-complete section* $A \subset X$ *with the following properties:*

(1) $T^i(A) \cap T^j(A) = \varnothing$ *for all* $0 \geq i, j < n$;
(2) $\mu\big(X \smallsetminus \bigcup_{i<n} T^i(A)\big) < \varepsilon$;
(3) $\mu(A) \leq 1/n$.

Proof. Let $(A_m)_{m\in\mathbb{N}}$ be a vanishing sequence of Borel markers for the equivalence relation E_T as given by Lemma 12.7. For every $x \in X$ and $m \in \mathbb{N}$ let

$$d_l(x, m) := \min\{\, k \,:\, T^{-k}(x) \in A_m \,\},$$

where the minimum over the empty set is set to be ∞. Similarly let

$$d_r(x, m) := \min\{\, k \,:\, T^{k}(x) \in A_m \,\}.$$

Consider now the set $B_m = \{x \in X : d_l(x, m) < n \text{ or } d_r(x, m) < n\, \}$. Since $\bigcap_{m\in\mathbb{N}} A_m = \varnothing$ we have that $\{B_m\}$ is also a decreasing sequence of Borel sets with empty intersection. Therefore we can fix m_0 large enough so that $\mu(B_{m_0}) < \varepsilon$. For every j among $\{0, 1, \ldots, n-1\}$ we define the set C_j of all $x \in X$ with $d_r(x, m_0) \geq n$ and with $d_l(x, m_0) = \mathrm{j} \bmod n$ if $d_l(x, m_0) \neq \infty$ or otherwise with $d_r(x, m_0) = \mathrm{j} \bmod n$. Notice that for every j the set C_j is Borel. For some $j_0 \in \{0, 1, \ldots, n-1\}$ we have that $\mu(C_{j_0}) \leq 1/n$. Let $A = C_{j_0}$. This is the desired A since by construction the T^i-translates of A are all disjoint for $0 \leq i < n$ and moreover $\bigcup_{0\leq i<n} T^i(A) \supseteq X \smallsetminus B_{m_0}$ with $\mu(B_{m_0}) < \varepsilon$. □

We have now the following theorem as consequence.

Theorem 2.8 (Rokhlin, Halmos)**.** *Let* $T \in \mathrm{APER}$. *Then for every* $n \geq 1$ *and for every* $\varepsilon > 0$ *there is a periodic* $S \in \mathrm{Aut}(\mu)$ *of period* n *so that* $\delta_u(T, S) < \varepsilon + 1/n$.

Proof. Let $A \subset X$ as in Lemma 2.7 and let $D = A \cup T(A) \cup \ldots \cup T^{n-1}(A)$. We define S as follows: for every $x \in D$ let $S(x) = T(x)$ and on the remaining "error set" D^c build an arbitrary n-periodic Borel automorphisms. A way to achieve this is to use the measure isomorphism theorem to identify (after re-normalizing the measure to 1) D^c with $[0,1]$ and then take S on D^c to be $(+\frac{1}{\mathrm{n}}) \mathrm{mod}\, 1$. $\square$

We have now as corollary the conjugacy theorem:

Theorem 2.9. *For $T \in \mathrm{Aut}(\mu)$ the following are equivalent:*
(i) $T \in \mathrm{APER}$;
(ii) $[T]_{\mathrm{conj}}$ is uniformly dense in APER, and
(iii) $[T]_{\mathrm{conj}}$ is weakly dense in $\mathrm{Aut}(\mu)$.

Proof. $(i \Rightarrow ii)$ Let $S \in \mathrm{APER}$ and let $\varepsilon > 0$. By Theorem 2.8 and for $N \in \mathbb{N}$ with $1/N < \varepsilon/4$ we can pick periodic S', T' both of period N so that $\delta_u(T, T'), \delta_u(S, S') < \varepsilon/4 + 1/N < \varepsilon/2$. By Lemma 2.6 we get $Q \in \mathrm{Aut}(\mu)$ with $QT'Q^{-1} = S'$. Hence

$$\delta_u(QTQ^{-1}, S) \leq \delta_u(QTQ^{-1}, QT'Q^{-1}) + \delta_u(QT'Q^{-1}, S) = \delta_u(QTQ^{-1}, QT'Q^{-1}) + \delta_u(S', S)$$

and since δ_u is left and right invariant we have

$$= \delta_u(QTQ^{-1}, QT'Q^{-1}) + \delta_u(S', S) = \delta_u(T, T') + \delta_u(S', S) < (\varepsilon/4 + 1/N) + (\varepsilon/4 + 1/N) < \varepsilon.$$

$(ii \Rightarrow iii)$ This follows from the fact that APER is weakly dense in $\mathrm{Aut}(\mu)$ and the fact that the uniform topology is stronger than the weak topology.
$(iii \Rightarrow i)$ Assume that $T \notin \mathrm{APER}$. Then for some $n \in \mathbb{N}$ we have that $\mu(\{x \in X : T^n(x) = x\}) = \varepsilon > 0$ and therefore $\delta_u(T^n, \mathrm{id}) \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Fix such n and ε and let $A = \{x \in X : T^n(x) = x\}$. Let also $D = [T]_{\mathrm{conj}}$. We will show that D is not weakly dense in APER. We already have that $D^n \subseteq \overline{B_u(id, 1-\varepsilon)}$. Notice moreover that $\overline{B_u(id, 1-\varepsilon)}$ is weakly closed. Since $p_n : g \mapsto g^n$ is continuous in every topological group, we have that $p_n^{-1}(\overline{B_u(id, 1-\varepsilon)})$ is still weakly closed, it contains D and it is disjoint from APER. Therefore $\overline{D}^w \subset p_n^{-1}(\overline{B_u(id, 1-\varepsilon)})$ completely misses APER. $\square$

2.D. **Ergodic automorphisms.**

3. Conjugary and unitary equivalence

Part 2. The spaces of actions of countable groups

Part 3. Appendix on descriptive set theory

4. Polish spaces

4.A. Definition and examples.

Definition 4.1. A topological space is called *Polish* if it is separable and completely metrizable (i.e. admits a complete compatible metric).

We work with Polish topological spaces as opposed to Polish metric spaces because we don't want to fix a particular complete metric, we may change it to serve different purposes; all we care about is that such a complete compatible metric exists. Besides, our maps are homeomorphisms and not isometries, so we work in the category of topological spaces and not metric spaces.

Examples 4.2.

(a) For all $n \in \mathbb{N}$, $n = \{0, 1, ..., n-1\}$ is Polish with discrete topology; so is $\mathbb{N}$;

(b) $\mathbb{R}^n$ and $\mathbb{C}^n$, for $n \geq 1$;

(c) Separable Banach spaces; in particular, separable Hilbert spaces, $\ell^p(\mathbb{N})$ and $L^p(\mathbb{R})$ for $0 < p < \infty$.

The following lemma, whose proof is left as an exercise, shows that when working with Polish spaces, we may always take a complete compatible metric $d \leq 1$:

Lemma 4.3. *If X is a topological space with a compatible metric d, then the following metric is also compatible: for $x, y \in X$, $D(x,y) = \min(d(x,y), 1)$.*

Proposition 4.4.
(a) Completion of any separable metric space is Polish.
(b) A closed subset of a Polish space is Polish (with respect to relative topology).
(c) A countable disjoint union[3] of Polish spaces is Polish.
(d) A countable product of Polish spaces is Polish (with respect to the product topology).

Proof. (a) and (b) are obvious. We leave (c) as an exercise and prove (d). To this end, let $X_n, n \in \mathbb{N}$ be Polish spaces and let $d_n \leq 1$ be a complete compatible metric for X_n. For $x, y \in \prod_{n \in \mathbb{N}} X_n$, define

$$d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} d_n(x(n), y(n)).$$

It is easy to verify that d is a complete compatible metric for the product topology on $\prod_{n \in \mathbb{N}} X_n$. $\square$

Examples 4.5.

(a) $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$;

[3] Disjoint union of topological spaces $\{X_i\}_{i \in I}$ is the space $\bigsqcup_{i \in I} X_i \coloneqq \bigcup_{i \in I} \{i\} \times X_i$ equipped with the topology generated by sets of the form $\{i\} \times U_i$, where $i \in I$ and $U_i \subseteq X_i$ is open.

(b) The Cantor space $\mathcal{C} := 2^{\mathbb{N}}$, with the discrete topology on 2;

(c) The Baire space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$, with the discrete topology on $\mathbb{N}$.

(d) The Hilbert cube $[0,1]^{\mathbb{N}}$.

We also obtain the following general class of examples as a nice application of Proposition 4.4. For a metric space (X,d), denote by $\mathrm{Iso}\big(X,d\big)$ the group of all isometries of (X,d). This is a group under composition and one easily checks that the pointwise convergence topology turns it into a topological group (i.e. multiplication and inverse are continuous).

Proposition 4.6. *Let (X,d) be a complete separable metric space. Then $\mathrm{Iso}(X,d)$ endowed with the pointwise convergence topology is Polish.*

Proof. Let $(x_n)_n \subseteq X$ be a dense sequence. Consider the map $\alpha : \mathrm{Iso}(X,d) \to X^{\mathbb{N}}$ by $T \mapsto \big(T(x_n)\big)_n$. It is easy to check that this is a topological embedding. Moreover, the image $Y := \alpha(\mathrm{Iso}(X,d))$ is a closed subset of $X^{\mathbb{N}}$: indeed,

$$Y = \left\{(y_n)_n \in X^{\mathbb{N}} : \forall n,m \quad d(x_n,x_m) = d(y_n,y_m)\right\}.$$

Since $X^{\mathbb{N}}$ is Polish, so is Y (being a closed subset), and hence $\mathrm{Iso}(X,d)$. □

By (b) of Proposition 4.4, closed subsets of Polish spaces are Polish. What other subsets have this property? The proposition below answers this question, but first we recall here that countable intersections of open sets are called G_δ sets, and countable unions of closed sets are called F_σ.

Lemma 4.7. *If X is a metric space, then closed sets are G_δ; equivalently, open sets are F_σ.*

Proof. Let $C \subseteq X$ be a closed set and let d be a metric for X. For $\varepsilon > 0$, define $U_\varepsilon = \{x \in X : d(x,C) < \varepsilon\}$, and we claim that $C = \bigcap_n U_{1/n}$. Indeed, $C \subseteq \bigcap_n U_{1/n}$ is trivial, and to show the other inclusion, fix $x \in \bigcap_n U_{1/n}$. Thus, for every n, we can pick $x_n \in C$ with $d(x,x_n) < 1/n$, so $x_n \to x$ as $n \to \infty$, and hence $x \in C$ by the virtue of C being closed. □

Proposition 4.8. *A subset of a Polish space is Polish if and only if it is G_δ.*

Proof. Let X be a Polish space and let d_X be a complete compatible metric on X.
$\Leftarrow$: We first prove that an open subset $U \subseteq X$ is Polish. The idea is to define a compatible metric for the topology of U so that it makes the boundary of U look like infinity (to prevent sequences that converge to boundary points from being Cauchy). It is easy to check that the following metric works: for $x,y \in U$,

$$d_U(x,y) = d_X(x,y) + \left|\frac{1}{d_X(x,\partial U)} - \frac{1}{d_X(y,\partial U)}\right|.$$

Now if $Y \subseteq X$ is G_δ, that is, $Y = \bigcap_{n\in\mathbb{N}} U_n$ with U_n open, then letting d_n be a complete compatible metric for U_n, we can define one for Y as follows: for $x,y,\in Y$,

$$d_Y(x,y) = \sum_{n\in\mathbb{N}} 2^{-n} d_n(x,y).$$

$\Rightarrow$ (Alexandrov): Let $Y \subseteq X$ be completely metrizable and let d_y be a complete compatible metric for Y. Define an open set $V_n \subseteq X$ as the union of all open sets $U \subseteq X$ that satisfy

(i) $U \cap Y \neq \varnothing$,
(ii) $\mathrm{diam}_{d_X}(U) < 1/n$,

(iii) $\operatorname{diam}_{d_Y}(U \cap Y) < 1/n$.

We show that $Y = \bigcap_{n \in \mathbb{N}} V_n$. First fix $x \in Y$ and take any $n \in \mathbb{N}$. Take an open neighborhood $U_1 \subseteq Y$ of x in Y of d_Y-diameter less than $1/n$. By the definition of relative topology, there is an open set U_2 in X such that $U_2 \cap Y = U_1$. Let U_3 be an open neighborhood of x in X of d_X-diameter less than $1/n$. Then $U = U_2 \cap U_3$ satisfies all of the conditions above. Hence $x \in V_n$.

Conversely, if $x \in \bigcap_{n \in \mathbb{N}} V_n$, then for each $n \in \mathbb{N}$, there is an open (relative to X) neighborhood $U_n \subseteq X$ of x satisfying the conditions above. Condition (ii) implies that $x \in \overline{Y}$, so any open neighborhood of x has a nonempty intersection with Y; because of this, we can replace U_n by $\bigcap_{m \leq n} U_m$ and assume without loss of generality that $(U_n)_{n \in \mathbb{N}}$ is decreasing. Now, take $x_n \in U_n \cap Y$. Conditions (i) and (ii) imply that $(x_n)_{n \in \mathbb{N}}$ converges to x. Moreover, condition (iii) and the fact that $(U_n)_{n \in \mathbb{N}}$ is decreasing imply that $(x_n)_{n \in \mathbb{N}}$ is Cauchy with respect to d_Y. Thus, since d_Y is complete, $x_n \to x'$ for some $x' \in Y$. Because limit is unique in Hausdorff spaces, $x = x' \in Y$. □

As an example of a G_δ subset of a Polish space, we give the following proposition, whose proof is left to the reader.

Proposition 4.9. *The Cantor space $\mathcal{C}$ is homeomorphic to a closed subset of the Baire space $\mathcal{N}$, whereas $\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathcal{C}$.*

4.B. **Cantor space as the "smallest" uncountable Polish space.** Recall that a *limit point* of a topological space is a point that is not isolated. A space is *perfect* if all its points are limit points. If P is a subset of a topological space X, we call P *perfect* in X if P is closed and perfect in its relative topology. For example, R^n, $R^{\mathbb{N}}$, C^n, $C^{\mathbb{N}}$, $[0,1]^{\mathbb{N}}$, $\mathcal{C}$, $\mathcal{N}$ are perfect. Another example of a perfect space is $C(X)$, where X compact metrizable. Note that $\mathbb{Q}$ is perfect as a topological space, but it is not a perfect subset of $\mathbb{R}$ as it is not closed.

We now list some important results, whose proofs can be found in [Kec95] and [Tse13].

The following is perhaps the first theorem in descriptive set theory.

Perfect Set Theorem 4.10 (Cantor?)**.** *The Cantor space $\mathcal{C}$ embeds into any nonempty perfect Polish space.*

The last theorem shows, in particular, that perfect Polish spaces have cardinality at least continuum.

Example 4.11. The space $\mathbb{Q}$ with its usual topology (the relative topology of $\mathbb{R}$) is not Polish since it is a perfect topological space, yet countable.

Theorem 4.12 (Cantor–Bendixson)**.** *Let X be a Polish space. Then X can be uniquely written as $X = P \cup U$, with P a perfect subset of X and U countable open.*

Theorems 4.10 and 4.12 give:

Corollary 4.13. *The Cantor space $\mathcal{C}$ embeds into any uncountable Polish space.*

This last corollary says that Polish spaces satisfy the Continuum Hypothesis, i.e. there is no Polish space of cardinality strictly in between countable and continuum. Inspired by this, we (Cantor?) isolate the following property of subsets of Polish spaces:

Definition 4.14. For a Polish space X, say that a subset $A \subseteq X$ has the *perfect set property* (PSP) if it either it is countable or contains a homeomorphic copy of $\mathcal{C}$.

By the last corollary, Polish spaces themselves have the PSP and hence so the G_δ subsets of Polish spaces being themselves Polish in the relative topology. One may wonder how far we can push this (Borel, analytic, co-analytic, $\mathbf{\Sigma}^2_1$, etc.), and this question gets difficult quickly, in a very precise sense.

4.C. **Possible cardinalities for Polish spaces.** What cardinalities can Polish spaces attain? The following gives an upper bound:

Proposition 4.15. *Any separable first countable topological space X has cardinality at most continuum.*

Proof. Fixing a countable dense set D, we embed X into $D^{\mathbb{N}}$ by choosing, for each $x \in X$, a sequence $(d_n)_n \subseteq D$ that converges to x (this can be done by first countability). Thus, $|X| \leq |D^{\mathbb{N}}| \leq |\mathbb{N}^{\mathbb{N}}| \leq |(2^{\mathbb{N}})^{\mathbb{N}}| = |2^{\mathbb{N}\times\mathbb{N}}| = |2^{\mathbb{N}}|$. $\square$

Assuming the Continuum Hypothesis (CH), this shows that the possible cardinalities for Polish spaces are countable or continuum, but can we prove this without assuming CH? Yes, this is exactly what Corollary 4.13 implies: any uncountable Polish space has cardinality at least continuum. Thus, the possible Polish space cardinalities are: $0, 1, 2, ..., \aleph_0, 2^{\aleph_0}$.

4.D. **Baire space as the "largest" Polish space.** Just like Cantor space, Baire space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ occupies a very special and important place among all Polish spaces, and this is mainly due to the following theorem:

Theorem 4.16. *For any Polish space X, there is a closed set $C \subseteq \mathcal{N}$ and a continuous bijection $f : C \to X$.*

One can show that Baire space can be continuously contracted to any of its nonempty closed subsets (by contracting the tree $\mathbb{N}^{<\mathbb{N}}$ to a subtree), which, in conjunction with the above theorem, gives:

Corollary 4.17. *Any nonempty Polish space X is a continuous image of $\mathcal{N}$.*

5. Borel sets

5.A. **σ-algebras and measurable spaces.** Recall that an *algebra* $\mathcal{A}$ on a set X is a family of subsets of X containing $\varnothing$ and closed under complements and finite unions (hence also finite intersections). An algebra $\mathcal{A}$ on X is called a *σ-algebra* if it is closed under countable unions (hence also countable intersections). For a family $\mathcal{E}$ of subsets of X, let $\sigma(\mathcal{E})$ denote the smallest σ-algebra containing $\mathcal{E}$. We say that $\mathcal{E}$ generates the given σ-algebra $\mathcal{A}$ or that $\mathcal{E}$ is a generating set for $\mathcal{A}$ if $\sigma(\mathcal{E}) = \mathcal{A}$.

For a collection $\mathcal{E}$ of subsets of X, put $\sim \mathcal{E} = \{A^c : A \in \mathcal{E}\}$, where $A^c = X \setminus A$.

Proposition 5.1. *Let X be a set and $\varnothing \in \mathcal{E} \subseteq \mathscr{P}(X)$. Then $\sigma(\mathcal{E})$ is the smallest collection S of sets that contains $\mathcal{E}, \sim \mathcal{E}$, and is closed under countable unions and countable intersections.*

Proof. Put $S' = \{A \in S : A, A^c \in S\}$. Clearly, $S' \supseteq \mathcal{E}$ and it is trivially closed under complements. Because complement of a union is the intersection of complements, S' is also closed under countable unions, and thus is a σ-algebra. Hence, $\sigma(\mathcal{E}) \subseteq S' \subseteq S \subseteq \sigma(\mathcal{E})$. $\square$

Definition 5.2. A *measurable space* is a pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a σ-algebra on X. For measurable spaces $(X, \mathcal{S}), (Y, \mathcal{A})$, a map $f : X \to Y$ is called *measurable* if $f^{-1}(A) \in \mathcal{S}$ for each $A \in \mathcal{A}$.

For a topological space Y, let $\mathcal{B}(Y)$ denote the σ-algebra generated by all open sets and it is called the *Borel σ-algebra* of Y. The sets in $\mathcal{B}(Y)$ are called *Borel sets*. For a measurable space $(X,\mathcal{S})$, a map $f : X \to Y$ is called *measurable* if it is measurable as a map from $(X,\mathcal{S})$ to $(Y,\mathcal{B}(Y))$, i.e. the preimage of a Borel set is in $\mathcal{S}$. For topological spaces X, Y, a map $f : X \to Y$ is called *Borel* (or *Borel measurable*) if it is measurable as a map from $(X,\mathcal{B}(X))$ to Y, i.e. the preimages of Borel sets are Borel.

Proposition 5.3. *Let $(X,\mathcal{S}),(Y,\mathcal{A})$ be measurable spaces and let $\mathcal{F}$ be a generating set for $\mathcal{A}$. Then, a map $f : X \to Y$ is measurable if $f^{-1}(A) \in \mathcal{S}$ for every $A \in \mathcal{F}$. In particular, if Y is a topological space and $\mathcal{A} = \mathcal{B}(Y)$, then f is measurable if the preimage of every open set is in $\mathcal{S}$.*

Proof. It is easy to check that $\mathcal{A}' = \{A \in \mathcal{A} : f^{-1}(A) \in \mathcal{S}\}$ is a σ-algebra and contains $\mathcal{F}$. Thus, $\mathcal{A}' = \mathcal{A}$ and hence $f^{-1}(\mathcal{A}) \subseteq \mathcal{S}$. □

This proposition in particular implies that continuous functions are Borel.

5.B. **The stratification of Borel sets into a hierarchy.** Let X be a topological space. We will now define the hierarchy of the Borel subsets of X, i.e. the recursive construction of Borel sets level-by-level, starting from the open sets.

Let ω_1 denote the first uncountable ordinal, and for $1 \leq \xi < \omega_1$, define by transfinite recursion the classes $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ of subsets of X as follows:

$$\begin{aligned}
\mathbf{\Sigma}^0_1(X) &= \{U \subseteq X : U \text{ is open}\} \\
\mathbf{\Pi}^0_\xi(X) &= \sim \mathbf{\Sigma}^0_\xi(X) \\
\mathbf{\Sigma}^0_\xi(X) &= \left\{\bigcup_n A_n : A_n \in \mathbf{\Pi}^0_{\xi_n}(X), \xi_n < \xi, n \in \mathbb{N}\right\}, \text{ if } \xi > 1.
\end{aligned}$$

In addition, we define the so-called *ambiguous classes* $\mathbf{\Delta}^0_\xi(X)$ by

$$\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X).$$

Traditionally, one denotes by $G(X)$ the class of open subsets of X, and by $F(X)$ the class of closed subsets of X. For any collection $\mathcal{E}$ of subsets of subsets of X, let

$$\begin{aligned}
\mathcal{E}_\sigma &= \left\{\bigcup_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N}\right\} \\
\mathcal{E}_\delta &= \left\{\bigcap_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N}\right\}.
\end{aligned}$$

Then we have $\mathbf{\Sigma}^0_1 = G(X), \mathbf{\Pi}^0_1(X) = F(X), \mathbf{\Sigma}^0_2(X) = F_\sigma(X), \mathbf{\Pi}^0_2(X) = G_\delta(X), \mathbf{\Sigma}^0_3(X) = F_{\sigma\delta}(X), \mathbf{\Pi}^0_3(X) = G_{\delta\sigma}(X)$, etc. Also, note that $\mathbf{\Delta}^0_1(X) = \{A \subseteq X : A \text{ is clopen}\}$.

Proposition 5.4 (Closure properties)**.** *For a topological space X and for each $\xi \geq 1$, the classes $\mathbf{\Sigma}^0_\xi(X), \mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X)$ are closed under finite intersections and finite unions. Moreover, $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, $\mathbf{\Pi}^0_\xi$ under countable intersections, and $\mathbf{\Delta}^0_\xi$ under complements.*

Proof. The only statement worth checking is the closedness of the classes $\mathbf{\Sigma}^0_\xi$ under finite intersections, but it easily follows by induction on ξ using the fact that

$$\bigcup_n A_n \cap \bigcup_n B_n = \bigcup_{n,m} (A_n \cap B_m).$$

The statements about $\boldsymbol{\Pi}^0_\xi$ follows from those about $\boldsymbol{\Sigma}^0_\xi$ by taking complements. □

Proposition 5.5. *Let X be a metrizable space.*
(a) $\boldsymbol{\Sigma}^0_\xi(X) \cup \boldsymbol{\Pi}^0_\xi(X) \subseteq \boldsymbol{\Delta}^0_{\xi+1}(X)$.
(b) $\mathcal{B}(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Delta}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Pi}^0_\xi(X)$.

Proof. We omit the proof as it is a tedious transfinite induction argument. We remark, however, that the metrizability assumption is only used in the proof of the base case of (a), namely, to show that $\boldsymbol{\Sigma}^0_1(X) \subseteq \boldsymbol{\Sigma}^0_2(X)$, which is the statement of Lemma 4.7. □

Thus, we have the following picture:

$$\begin{tikzcd}[column sep=small]
 & \boldsymbol{\Sigma}^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Sigma}^0_2 \arrow[dr, phantom, sloped, "\subseteq"] & & & & \boldsymbol{\Sigma}^0_\xi \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\boldsymbol{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Delta}^0_3 & \cdots & \boldsymbol{\Delta}^0_\xi \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Delta}^0_{\xi+1} & \cdots \\
 & \boldsymbol{\Pi}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \boldsymbol{\Pi}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] & & & & \boldsymbol{\Pi}^0_\xi \arrow[ur, phantom, sloped, "\subseteq"] & &
\end{tikzcd}$$

$$\underbrace{\hphantom{\boldsymbol{\Delta}^0_1 \quad \boldsymbol{\Delta}^0_2 \quad \boldsymbol{\Delta}^0_3 \quad \cdots \quad \boldsymbol{\Delta}^0_\xi \quad \boldsymbol{\Delta}^0_{\xi+1} \quad \cdots}}_{\mathcal{B}}$$

Note that if X is second countable, then $|\boldsymbol{\Sigma}^0_1(X)| \leq 2^{\aleph_0}$ and hence, by induction on $\xi < \omega_1$, $|\boldsymbol{\Sigma}^0_\xi(X)| \leq |(2^{\aleph_0})^{\aleph_0}| = |2^{\aleph_0 \times \aleph_0}| = 2^{\aleph_0}$, hence also $|\boldsymbol{\Pi}^0_\xi(X)| \leq 2^{\aleph_0}$. Thus, it follows from (b) of the previous proposition that $|\mathcal{B}(X)| \leq |\omega_1 \times 2^{\aleph_0}|$, and by Axiom of Choice, $|\omega_1 \times 2^{\aleph_0}| = 2^{\aleph_0}$, so there are at most continuum many Borel sets.

Example 5.6. Let C^1 be the set of all continuously differentiable function in $C([0,1])$ (at the endpoints we take one-sided derivatives). We will show that C^1 is $\boldsymbol{\Pi}^0_3$ and hence Borel.

It is not hard to check that for $f \in C([0,1])$, $f \in C^1$ iff for all $\varepsilon \in \mathbb{Q}^+$ there exist rational open intervals $I_0, ..., I_{n-1}$ covering $[0,1]$ such that for all $j < n$:

$$\forall a,b,c,d \in I_j \cap [0,1] \text{ with } a \neq b, c \neq d \left(\left| \frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(d)}{c-d} \right| \leq \varepsilon \right).$$

So if for an open interval J and $\varepsilon > 0$, we put

$$A_{J,e} = \left\{ f \in C([0,1]) : \forall a,b,c,d \in J \cap [0,1] \text{ with } a \neq b, c \neq d, \left| \frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(d)}{c-d} \right| \leq \varepsilon \right\},$$

we have that $A_{j,e}$ is closed in $C([0,1])$ and

$$C^1 = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_n \bigcup_{(I_0,...,I_{n-1})} \bigcap_{j<n} A_{I_j,\varepsilon},$$

where $(I_0, ..., I_{n-1})$ varies over all n-tuples of rational open intervals with $\bigcup_{i<n} I_i \supseteq [0,1]$. Thus, C^1 is $\boldsymbol{\Pi}^0_3$.

5.C. **Universal sets for $\boldsymbol{\Sigma}^0_\xi$ and $\boldsymbol{\Pi}^0_\xi$, and diagonalization.** The classes $\boldsymbol{\Sigma}^0_\xi, \boldsymbol{\Pi}^0_\xi$ and $\boldsymbol{\Delta}^0_\xi$ provide for each Polish space X a hierarchy for $\mathcal{B}(X)$ of at most ω_1 levels. We will next show that this is indeed a proper hierarchy, i.e., all these classes are distinct, when X is uncountable. To accomplish this, we will construct, for each $\xi < \omega_1$, a set that parameterizes $\boldsymbol{\Sigma}^0_\xi$ and then apply the Cantor diagonalization technique.

Definition 5.7. Let $\boldsymbol{\Gamma}$ be a class of sets in topological spaces (such as $\boldsymbol{\Sigma}^0_\xi, \boldsymbol{\Pi}^0_\xi$, $\boldsymbol{\Delta}^0_\xi$, $\mathcal{B}$, etc.) and let X, Y be topological spaces. We say that a set $U \subseteq Y \times X$ *parameterizes* $\boldsymbol{\Gamma}(X)$ if

$$\{U_y : y \in Y\} = \boldsymbol{\Gamma}(X).$$

If, moreover, U itself is in $\mathbf{\Gamma}$ (i.e. $U \in \mathbf{\Gamma}(Y \times X)$), we say that U is *Y-universal for $\mathbf{\Gamma}(X)$.*

Theorem 5.8. *Let X be a separable metrizable space. Then for each $\xi \geq 1$, there is a $\mathcal{C}$-universal set for $\mathbf{\Sigma}^0_\xi(X)$, and similarly for $\mathbf{\Pi}^0_\xi(X)$.*

Proof. We prove by induction on ξ. Let $(V_n)_{n \in \mathbb{N}}$ be an open basis for X. Because every $\mathbf{\Sigma}^0_1$(=open) set is a union of some subsequence of these V_n, we define $U \subseteq \mathcal{C} \times X$ as follows: for $y \in \mathcal{C}$, put

$$U_y = \bigcup_{n, y(n)=1} V_n.$$

It is clear U parameterizes $\mathbf{\Sigma}^0_1(X)$. Moreover, U is open because for $(y, x) \in \mathcal{C} \times X$,

$$(y, x) \in U \iff x \in \bigcup_{n, y(n)=1} V_n \iff \exists n \in \mathbb{N}\ (y(n) = 1 \wedge x \in V_n).$$

Thus indeed, U is $\mathcal{C}$-universal for $\mathbf{\Sigma}^0_1$.

Note next that if $U \subseteq \mathcal{C} \times X$ is $\mathcal{C}$-universal for $\mathbf{\Gamma}(X)$, then U^c is $\mathcal{C}$-universal for the dual class $\check{\mathbf{\Gamma}}(X)$. In particular, if there is a $\mathcal{C}$-universal set for $\mathbf{\Sigma}^0_\xi(X)$, there is also one for $\mathbf{\Pi}^0_\xi(X)$.

We skip the argument for handling countable unions and just remark that we encode countably many $\mathcal{C}$-parameters into elements of $\mathcal{C}^{\mathbb{N}}$, and the latter space is easily shown to be homeomorphic to $\mathcal{C}$ using a bijection $\mathbb{N} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ (just like when showing that $\mathbb{Q}$ is countable). ∎

Corollary 5.9. *Let X be separable metrizable and Y be uncountable Polish. For any $1 \leq \xi < \omega_1$, there is a Y-universal set for $\mathbf{\Sigma}^0_\xi(X)$, and similarly for $\mathbf{\Pi}^0_\xi$.*

Proof. We skip the details and just remark that this is done using the fact that Cantor space embeds into any uncountable Polish space (Corollary 4.13). ∎

Lemma 5.10 (Diagonalization)**.** *For a set X and $R \subseteq X^2$, put* $\mathrm{AntiDiag}(R) = \{x \in X : \neg R(x, x)\}$. *Then* $\mathrm{AntiDiag}(R) \neq R_x$ *for any $x \in X$.*

Proof. Assume for contradiction that $\mathrm{AntiDiag}(R) = R_x$, for some $x \in X$. Then we get a contradiction because

$$\neg R(x, x) \iff x \in \mathrm{AntiDiag}(R) \iff x \in R_x \iff R(x, x).$$

∎

Corollary 5.11. *For every uncountable Polish space X and every $1 \leq \xi < \omega_1$, $\mathbf{\Sigma}^0_\xi(X) \neq \mathbf{\Pi}^0_\xi(X)$. In particular, $\mathbf{\Delta}^0_\xi(X) \subsetneq \mathbf{\Sigma}^0_\xi(X) \subsetneq \mathbf{\Delta}^0_{\xi+1}(X)$, and the same holds for $\mathbf{\Pi}^0_\xi$.*

Proof. Let $U \subseteq X \times X$ be an X-universal set for $\mathbf{\Sigma}^0_\xi(X)$ and take $A = \mathrm{AntiDiag}(U)$. Since $A = \delta^{-1}(U^c)$, where $\delta : X \to X^2$ by $x \mapsto (x, x)$, $A \in \mathbf{\Pi}^0_\xi(X)$. However, by the Diagonalization lemma, $A \neq U_x$ for any $x \in X$, and thus $A \notin \mathbf{\Sigma}^0_\xi(X)$. ∎

5.D. **Turning Borel sets into clopen sets.** The following theorem is truly one of the most useful facts about Borel sets. Recall that a Polish space X is formally a set X with a topology $\mathcal{T}$ on it (i.e. the collection of the open sets), so it is really a pair $(X, \mathcal{T})$. We denote the Borel subsets of X by $\mathcal{B}(X, \mathcal{T})$ or just $\mathcal{B}(\mathcal{T})$, when we want to emphasize the topology with respect to which the Borel sets are taken.

Theorem 5.12. *Let $(X,\mathcal{T})$ be a Polish space. For any countable collection of Borel sets $\{A_n\}_n \subseteq X$, there is a finer Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ with respect to which each A_n is clopen, yet $\mathcal{B}(\mathcal{T}') = \mathcal{B}(\mathcal{T})$.*

We now give a couple of very useful applications.

Corollary 5.13. *Borel subsets of Polish spaces have the PSP.*

Proof. Let B be an uncountable Borel subset of a Polish space $(X,\mathcal{T})$. By the previous theorem, there is a Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ in which B is clopen and hence $(B,\mathcal{T}'|_B)$ is Polish, where $\mathcal{T}'|_B$ denotes the relative topology on B with respect to $\mathcal{T}'$. Now by the PSP for Polish spaces, there is an embedding $f : \mathcal{C} \hookrightarrow (B,\mathcal{T}'|_B)$. But then f is still continuous as a map from $\mathcal{C}$ into $(B,\mathcal{T}|_B)$ as $\mathcal{T}|_B$ has fewer open sets. Hence, because $\mathcal{C}$ is compact, f is still automatically an embedding from $\mathcal{C}$ into $(B,\mathcal{T}|_B)$. ☐

Corollary 5.14. *Let $(X,\mathcal{T})$ be a Polish space, Y be a second countable space, and $f : X \to Y$ be a Borel function. There is a Polish topology $\mathcal{T}_f \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_f) = \mathcal{B}(T)$ that makes f continuous.*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis for Y and let $\mathcal{T}_f \supseteq \mathcal{T}$ be a Polish topology on X that makes each $f^{-1}(V_n)$ clopen and has the same Borel sets as $\mathcal{T}$. The function $f : (X,\mathcal{T}_f) \to Y$ is now continuous. ☐

Corollary 5.15. *Let $(X,\mathcal{T})$ be a Polish space and $B \subseteq X$ be Borel. There is a closed subset $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \to A$. In particular, if $A \neq \varnothing$, there is a continuous surjection $\bar{f} : \mathcal{N} \twoheadrightarrow A$.*

Proof. Let $\mathcal{T}' \supseteq \mathcal{T}$ be a Polish topology making B clopen. Hence $(B,\mathcal{T}'|_B)$ is Polish, so we can apply Theorem 4.16 and Corollary 4.17. ☐

Corollary 5.16. *For any Polish $(X,\mathcal{T})$, there is a zero-dimensional Polish topology $T_0 \supseteq T$ with $\mathcal{B}(T_0) = \mathcal{B}(T)$.*

Proof. Left as an exercise (iterate Theorem 5.12). ☐

Corollary 5.17. *Any Borel action of a countable group Γ on a Polish space $(X,\mathcal{T})$ has a continuous realization, i.e. there is a Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}') = \mathcal{B}(\mathcal{T})$, with respect to which the action is continuous. In fact, $\mathcal{T}'$ can be taken to be zero-dimensional.*

Proof. For the first statement, fix a countable basis $\mathcal{U}$ for $\mathcal{T}$ and take a Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ that makes the sets $\gamma^{-1}U$, $\gamma \in \Gamma, U \in \mathcal{U}$, clopen. Using the associativity of the action, one easily verifies that the action is continuous with respect to $\mathcal{T}'$. To also get zero-dimensionality, one has to alternate the latter procedure and Corollary 5.16 countably many times. ☐

5.E. **Images under small-to-one Borel functions.** We will see in a later section that continuous images of Borel (or even just closed) sets may no longer be Borel (they are called *analytic*). However, the situation may be different if the preimage of every point is "small". In this subsection, we will list some results with various notions of "small", starting from the cases where the domain space itself if "small".

Below we use the terms *σ-compact* or K_σ for subsets of topological spaces that are countable unions of compact sets. Also, for topological spaces X,Y, let $\mathrm{proj}_X : X \times Y \to X$ denote the projection function onto the X coordinate.

Proposition 5.18. (*a*) *Continuous functions map compact sets to compact sets.*

(*b*) *Continuous functions map K_σ sets to K_σ sets.*

(*c*) **Tube lemma.** *For topological spaces X, Y with Y compact,* proj_X *maps closed subsets of $X \times Y$ to closed subsets of X.*

Proof. (a) is just by unraveling the definitions and it immediately implies (b). For (c), let $F \subseteq X \times Y$ be closed, $x \notin \mathrm{proj}_X(F)$ and consider the open cover $(V_y)_{y \in Y}$ of Y where $V_y \ni y$ is open and is such that for some nonempty open neighborhood $U_y \subseteq X$ of x, $U_y \times V_y$ is disjoint from F. ☐

Examples 5.19.

(a) $\mathrm{proj}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ does not in general map closed sets to closed sets: e.g., take F to be the graph of $1/x$ with domain $(0, 1]$, then F is closed, but its projection is $(0, 1]$.

(b) However, because $\mathbb{R}$ is σ-compact (hence $K_\sigma = F_\sigma$) and Hausdorff (hence compact sets are closed), it follows from (b) of Proposition 5.18 that $\mathrm{proj}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ maps F_σ sets (in particular, closed sets) to F_σ sets.

The following is one of the most used results in descriptive set theory.

Theorem 5.20 (Luzin–Souslin)**.** *Let X, Y be Polish spaces and $f : Y \to X$ be Borel. If $A \subseteq Y$ is Borel and $f\downharpoonright_A$ is injective, then $f(A)$ is Borel.*

Corollary 5.21. *Let X, Y be Polish and $f : X \to Y$ be Borel. If f is injective, then it is a Borel embedding, i.e. f maps Borel sets to Borel sets.*

The Luzin–Souslin theorem together with Corollary 5.15 gives the following characterization of Borel sets:

Corollary 5.22. *A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$.*

This shows the contrast between Borel and analytic as the latter sets are just continuous images of closed subsets of $\mathcal{N}$.

Now, how big can the "small" be so that the Borel sets are still closed under "small"-to-one images? It turns out that for small being σ-compact, this is still true and this is a deep theorem of Arsenin and Kunugui [Kec95, 18.18]. Here we will only state a very important special case of this, which will be enough for our purposes.

For topological spaces X, Y, call a set $A \subseteq X \times Y$ a *Borel graph* if for every $x \in X$, the fiber $A_x := \{y \in Y : (x, y) \in A\}$ has at most one element.

Theorem 5.23 (Luzin–Novikov)**.** *Let X, Y be Polish spaces and $B \subseteq X \times Y$ be a Borel set all of whose X-fibers are countable, i.e. for every $x \in X$, B_x is countable. Then B can be partitioned into countably many disjoint Borel graphs $B = \bigsqcup_n B_n$.*

Corollary 5.24. *The class of Borel subsets of Polish spaces is closed under countable-to-one Borel images.*

Proof. Let Z, X be Polish spaces, $f : Z \to X$ be a countable-to-one Borel function, $B \subseteq Z$ a Borel set, and we show that $f(B)$ is Borel. By replacing B with $\mathrm{graph}(f\downharpoonright_B)$, we may assume that $Z = X \times Y$, for some Polish space Y, and $f = \mathrm{proj}_X$. By the Luzin–Novikov theorem, $B = \dot{\bigcup}_n B_n$, where each B_n is a Borel graph. For each n, $f(B_n)$ is Borel by Theorem 5.20, and thus, so is $f(B) = \bigcup_n f(B_n)$. □

The next corollary says, in particular, that given a Borel set $B \subseteq X \times Y$ with countable X-fibers, for each $x \in \mathrm{proj}_X(B)$, we can choose *in a Borel way* ("uniformly") a witness $y \in Y$ with $(x, y) \in B$.

Corollary 5.25 (Countable uniformization)**.** *For Polish spaces Z, X, any countable-to-one Borel function $f : Z \to X$ admits a Borel right inverse $g : f(Z) \to Z$.*

Proof. Just like in the proof of Corollary 5.24, we may assume that $Z = X \times Y$ and $f = \mathrm{proj}_X$. By the Luzin–Novikov theorem, $B = \dot{\bigcup}_n B_n$, where each B_n is a Borel graph. Define $k : X \to \mathbb{N}$ by $x \mapsto$ the least $n \in \mathbb{N}$ with $x \in \mathrm{proj}_X(B_n)$, and finally define $g : X \to X \times Y$ by $x \mapsto (x, y)$, where $y \in Y$ is the unique element with $(x, y) \in B_{k(x)}$. It is straightforward to check that the function k, and hence also g, is Borel. □

5.F. **The Borel Isomorphism Theorem.** The following is the Borel analogue of the Schröder–Bernstein–Cantor Theorem, which states that if one set can be injected into another and vice versa, then there is a bijection between them. It is yet another corollary of the fact that one-to-one Borel images of Borel sets are Borel.

Corollary 5.26 (The Borel Schröder–Bernstein–Cantor Theorem)**.** *Let X, Y be Polish and $f : X \hookrightarrow Y$, $g : Y \hookrightarrow X$ be Borel injections. Then X and Y are Borel isomorphic.*

Proof. Run the same proof as for the regular Schröder–Bernstein theorem and note that all the sets involved are images of Borel sets under f or g, and hence are themselves Borel by **??**. Thus, the resulting bijection is a Borel isomorphism. □

The following theorem shows how robust the framework of Polish spaces is when studying Borel sets and beyond.

The Borel Isomorphism Theorem 5.27. *Any two Polish spaces of the same cardinality are isomorphic. In particular, any two uncountable Polish spaces are Borel isomorphic.*

Proof. The statement for countable Polish spaces is obvious since their Borel σ-algebra is all of their powerset. For uncountable Polish space, it is enough to show that if X is uncountable, then it is Borel isomorphic to $\mathcal{C}$. By the Borel Schröder–Bernstein–Cantor theorem, it is enough to show that there are Borel injections $\mathcal{C} \hookrightarrow X$ and $X \hookrightarrow \mathcal{N}$ since $\mathcal{N}$ embeds into $\mathcal{C}$. The embedding $\mathcal{C} \hookrightarrow X$ is given by Corollary 4.13. By Theorem 4.16, there is a closed subset $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \to X$. By **??**, f maps Borel sets to Borel sets, so f^{-1} is a Borel map, embedding X into $\mathcal{N}$. □

Corollary 5.28. *Any Polish space admits a Borel linear ordering.*

Proof. By the Borel Isomorphism Theorem, any Polish space X is either Borel isomorphic to $\mathbb{R}$ or embeds into $\mathbb{N}$, so we can copy the natural linear orderings from the latter spaces. □

5.G. **Standard Borel spaces.** As the Borel Isomorphism Theorem shows, it really does not matter which Polish space to consider when working in the Borel context. The following definition makes abstracting from the topology but keeping the Borel structure precise.

Definition 5.29. A measurable space $(X, \mathcal{S})$ is called a standard Borel space if there is a Polish topology $\mathcal{T}$ on X such that $\mathcal{B}(\mathcal{T}) = \mathcal{S}$. In this case, we call $\mathcal{T}$ a compatible Polish topology and refer to the sets in $\mathcal{S}$ as Borel sets.

Examples 5.30.

(a) An obvious example of a standard Borel space is a Polish space with its Borel σ-algebra: $(X, \mathcal{B}(X))$.

(b) A less immediate example, due to Theorem 5.12, is a Borel subset A of a Polish space X equipped with the relative Borel σ-algebra: $(A, \mathcal{B}(X)\downharpoonright_A)$, where $\mathcal{B}(X)\downharpoonright_A = \{B \cap A : B \in \mathcal{B}(X)\} = \{B \in \mathcal{B}(X) : B \subseteq A\}$.

6. Analytic sets

It is clear that the class of Borel sets is closed under continuous preimages, but is it closed under continuous images?

Definition 6.1. A subset A of a Polish space X is called analytic if it is a continuous image of a Borel subset of some Polish space; more precisely, if there is a Polish space Y, a Borel set $B \subseteq Y$ and a continuous function $f : Y \to X$ such that $f(B) = A$.

Clearly, all Borel sets are analytic, but is the converse true? Historically, Lebesgue had a "proof" that continuous images of Borel sets are Borel, but several years later Souslin found a mistake in Lebesgue's proof; moreover, he constructed an example of a closed set whose projection was not Borel. Hence continuous images of Borel sets were new kinds of sets, which he and his advisor Luzin called analytic and systematically studied the properties thereof. This is often considered the birth of descriptive set theory.

6.A. **Basic facts and closure properties.** Before we exhibit an analytic set that is not Borel, we give the following equivalences to being analytic.

Proposition 6.2. *Let X be Polish and $A \subseteq X$. The following are equivalent:*

(1) A is analytic;
(2) There is Polish Y and continuous $f : Y \to X$ with $A = f(Y)$;
(3) There is continuous $f : \mathcal{N} \to X$ with $A = f(\mathcal{N})$;
(4) There is closed $F \subseteq X \times \mathcal{N}$ with $A = \mathrm{proj}_X(F)$;
(5) There is Polish Y and Borel $B \subseteq X \times Y$ with $A = \mathrm{proj}_X(B)$;
(6) There is Polish Y, Borel $B \subseteq Y$ and Borel $f : Y \to X$ with $A = f(B)$;

Proof. (4)⇒(5)⇒(1)⇒(6) are trivial, (1)⇒(2) is immediate from Theorem 5.12, (2)⇒(3) follows from Corollary 4.17, and (3)⇒(4) follows from the fact that graphs of continuous functions are closed and $f(Y) = \mathrm{proj}_X(\mathrm{graph}(f))$.

Finally, the implication (6)⇒(1) follows from Corollary 5.14. Alternatively, one could deduce (6)⇒(5) from the fact that if $f : Y \to X$ and $B \subseteq Y$ are Borel, then $\mathrm{graph}(f\downharpoonright_B)$ is Borel and $f(B) = \mathrm{proj}_X(\mathrm{graph}(f\downharpoonright_B))$. □

Let $\mathbf{\Sigma}_1^1$ denote the class of all analytic subsets of Polish spaces, so for a Polish space X, $\mathbf{\Sigma}_1^1(X)$ is the set of all analytic subsets of X.

Proposition 6.3 (Closure properties of $\mathbf{\Sigma}_1^1$)**.** *The class $\mathbf{\Sigma}_1^1$ is closed under*

(i) continuous images and preimages;
(ii) (in fact) Borel images and preimages;
(iii) countable intersections and unions.

Proof. We only prove the closure under countable intersections and leave the rest as an exercise. Let A_n be analytic subsets of a Polish space X. By (4) of Proposition 6.2, there are closed sets $C_n \subseteq X \times \mathcal{N}$ such that $A_n = \operatorname{proj}_X(C_n)$. Let $Y = X \times \mathcal{N}^\infty$ and consider the set $C \subseteq Y$ defined by

$$(x, (y_n)_{n\in\mathbb{N}}) \in Y \iff \forall n \in \mathbb{N}(x, y_n) \in C_n.$$

Clearly, C is Borel (in fact it is closed) and $\bigcap_n A_n = \operatorname{proj}_X(C)$. ☐

Let $\mathbf{\Pi}_1^1 =\sim \mathbf{\Sigma}_1^1$ denote the dual class, and we call the elements of $\mathbf{\Pi}_1^1$ co-analytic. By (4) of Proposition 6.2, we have

$$\mathbf{\Sigma}_1^1 = \exists^{\mathcal{N}}\mathcal{B} = \exists^{\mathcal{N}}\mathbf{\Pi}_1^0,$$

and consequently,

$$\mathbf{\Pi}_1^1 = \forall^{\mathcal{N}}\mathcal{B} = \forall^{\mathcal{N}}\mathbf{\Sigma}_1^0.$$

Furthermore, put $\mathbf{\Delta}_1^1 = \mathbf{\Sigma}_1^1 \cap \mathbf{\Pi}_1^1$. It is clear that $\mathcal{B} \subseteq \mathbf{\Delta}_1^1$, and we will see below that these are actually equal.

6.B. **A universal set for $\mathbf{\Sigma}_1^1$.** We now focus on showing that $\mathbf{\Sigma}_1^1 \neq \mathbf{\Pi}_1^1$ and hence there are analytic sets that are not Borel. As with the Borel hierarchy, we start with a universal analytic set:

Theorem 6.4 (Souslin)**.** *For any uncountable Polish Y and Polish X, there is an Y-universal set $U \subseteq Y \times X$ for $\mathbf{\Sigma}_1^1(X)$. The same holds for $\mathbf{\Pi}_1^1(X)$.*

Proof. The idea is to use (4) of Proposition 6.2, so we start with a Y-universal set $F \subseteq Y \times (X \times \mathcal{N})$ for $\mathbf{\Pi}_1^0(X \times \mathcal{N})$, which exists by Corollary 5.9. Put $U = \operatorname{proj}_{Y\times X}(F) = \{(y,x) \in Y \times X : \exists z \in \mathcal{N}(y,x,z) \in F\}$ and note that U is analytic being a projection of a closed set. We claim that U also parametrizes $\mathbf{\Sigma}_1^1(X)$. Indeed, let $A \subseteq X$ be analytic, so by (4) of Proposition 6.2, there is a closed set $C \subseteq X \times \mathcal{N}$ with $A = \operatorname{proj}_X(C)$. Then there is $y \in Y$ with $F_y = C$ and hence $A = \operatorname{proj}_X(C) = \operatorname{proj}_X(F_y) = (\operatorname{proj}_{Y\times X}(F))_y = U_y$ and we are done. ☐

Corollary 6.5 (Souslin)**.** *For any uncountable Polish space X, $\mathbf{\Sigma}_1^1(X) \neq \mathbf{\Pi}_1^1(X)$. In particular, $\mathcal{B}(X) \subseteq \mathbf{\Delta}_1^1(X) \neq \mathbf{\Sigma}_1^1(X)$, and same for $\mathbf{\Pi}_1^1(X)$.*

Proof. Take an X-universal set $U \subseteq X\times X$ for $\mathbf{\Sigma}_1^1(X)$ and put $A = \operatorname{AntiDiag}(U) = \{x \in X : (x,x) \notin U\}$. Let $\delta : X \to X \times X$ by $x \mapsto (x,x)$ and note that it is continuous. Because $A = \delta^{-1}(U^c)$ and U^c is co-analytic, A is also co-analytic. However, it is not analytic since otherwise A would have to be equal to a fiber U_x of U, for some $x \in X$, contradicting the diagonalization lemma (Lemma 5.10).

In particular, A is not Borel, so A^c is analytic but not Borel. ☐

6.C. **Analytic separation and Borel = $\mathbf{\Delta}^1_1$.**

Theorem 6.6 (Luzin)**.** *Let X be a Polish space and let $A, B \subseteq X$ be disjoint analytic sets. There is a Borel set $C \subseteq X$ that separates A and B, i.e. $D \supseteq A$ and $D^c \supseteq B$.*

The following corollary is really what started the development of descriptive set theory.

Corollary 6.7 (Souslin)**.** *Let X be Polish and $A \subseteq X$. If A and A^c are both analytic, then A is Borel. In other words, $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$.*

Proof. Take a Borel set B separating A and A^c and note that B has to be equal to A. ☐

Here is an example to illustrate the usefulness of this.

Corollary 6.8. *Let X, Y be Polish and $f : X \to Y$. The following are equivalent:*
(1) f is Borel;
(2) The graph of f is Borel;
(3) The graph of f is analytic.

Proof. (1)⇒(2): Fix a countable basis $\{V_n\}_{n\in\mathbb{N}}$ for Y and note that for $(x, y) \in X \times Y$, we have
$$f(x) = y \iff \forall n (y \in V_n \to x \in f^{-1}(V_n)).$$
Thus
$$\operatorname{graph}(f) = \bigcap_n \left(\operatorname{proj}_Y^{-1}(V_n^c) \cup \operatorname{proj}_X^{-1}(f^{-1}(V_n))\right),$$
and hence is Borel.
(3)⇒(1): Assume (3) and let $U \subseteq Y$ be open; we need to show that $f^{-1}(U)$ is Borel. But for $x \in X$, we have
$$\begin{aligned} x \in f^{-1}(U) &\iff \exists y \in Y (f(x) = y \text{ and } y \in U) \\ &\iff \forall y \in Y (f(x) = y \to y \in U), \end{aligned}$$
so $f^{-1}(U)$ is both analytic and co-analytic, and hence is Borel by Souslin's theorem. ☐

Corollary 6.9. *Let X be Polish and let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of analytic subsets of X. Then there is a disjoint family $\{B_n\}_{n\in\mathbb{N}}$ of Borel sets with $B_n \supseteq A_n$.*

7. Borel measures and the measure algebra

7.A. **Definitions and examples.** Let X be a Polish space and let $\mathcal{B}(X)$ denote the Borel σ-algebra of X.

Definition 7.1. A Borel measure on X is a function $\mu : \mathcal{B}(X) \to \mathbb{R}^+ \cup \{\infty\}$ that takes $\varnothing$ to 0 and that is countably additive, i.e. for pairwise disjoint Borel sets $A_n, n \in \mathbb{N}$, we have
$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$

Examples 7.2.

(a) The Lebesgue measure on $\mathbb{R}^n$ defined first on rectangles as the product of their side lengths, and then extended to all Borel sets using Caratheodory's extension theorem. On $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$, this measure corresponds to our intuition of what length, area, and volume of sets should be.

(b) The natural measure on the unit circle S^1 defined by pushing forward the measure from $[0,1]$ to S^1 via the map $x \mapsto e^{2\pi x i}$.

(c) The Cantor space can be equipped with the so-called coin flip measure, which is given by $\mu(N_s) = 2^{-|s|}$, thus $\mu(\mathcal{C}) = 1$.

(d) In general, it is a theorem of Haar that every locally compact Hausdorff topological group admits a unique, up to a constant multiple, nontrivial regular left-invariant measure that is finite on compact sets; it is called Haar measure. This generalizes all of the above examples, including the coin flip measure on the Cantor space since we can identify $\mathcal{C} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

(e) On any set X, one can define the so-called counting measure μ_c by giving each singleton measure 1. Similarly, when X is say $\mathbb{N}^+$, one can also assign measure $1/2^n$ to $\{n\}$, for $n \in \mathbb{N}^+$ and obtain a probability measure.

A Borel measure μ on X is called *continuous* (or nonatomic) if every singleton has measure zero. For example, the measures in all but the last example above are continuous, whereas in the last example it is purely atomic.

Furthermore, call a Borel measure μ on X *finite* if $\mu(X) < \infty$, and it is called *σ-finite* if X can be written as $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$. In case $\mu(X) = 1$, we call μ a Borel *probability measure*. For example, the measures on S^1 and $\mathcal{C}$ defined above are probability measure, the Lebesgue measure on $\mathbb{R}^n$ is σ-finite, and the point measure on any uncountable set X is not σ-finite. In analysis and descriptive set theory, one usually deals with σ-finite measures, and even more often with probability measures.

Below, we denote by $\mathrm{P}(X)$ the set of all Borel probability measures on X.

7.B. **The null ideal and measurability.** Let X be a standard Borel space.

Definition 7.3. For a Borel measure μ on X, the *null ideal of* μ, noted NULL_μ, is the family of all subsets of Borel sets of measure 0.

Because of countable additivity, NULL_μ is a σ-ideal, and the sets in it are called μ-null (or just null) sets.

For two sets $A, B \subseteq X$, we write $A =_\mu B$ if $A \triangle B \in \mathrm{NULL}_\mu$. This is clearly an equivalence relation.

Definition 7.4. For a Borel measure μ, a set $A \subseteq X$ is called *μ-measurable* if $A =_\mu B$ for some Borel set B. In this case, we will define $\mu(A) = \mu(B)$, and extend μ to be defined on all measurable sets.

Clearly, μ-measurable subsets of X form a σ-algebra, which we denote by $\mathrm{MEAS}_\mu(X)$.

Definition 7.5. A subset A of a Polish space X is called *universally measurable* if it is μ-measurable for every σ-finite Borel measure μ.

In this definition, due to σ-finiteness of μ, the set A is μ-measurable if and only if $A \cap B$ is μ-measurable for every μ-measurable subset $B \subseteq X$ of finite μ-measure. This shows that we can replace "σ-finite" by "probability" in the definition of universal measurability. In fact, it is enough to consider continuous probability measures since probability measures can have at most countably many atoms and countable sets are clearly universally measurable.

It is clear that universally measurable subsets of X form a σ-algebra and we denote it by $\mathrm{MEAS}(X)$. By definition, Borel sets are universally measurable. How far can we push this? The following is a non-trivial theorem, whose proof we omit here.

Theorem 7.6. *Analytic (and hence also co-analytic) subsets of Polish spaces are universally measurable.*

This theorem might make one hopeful that perhaps all projective sets are universally measurable. However, it turns out that already for $\mathbf{\Sigma}^1_2$ subsets of $\mathbb{R}$ the question of whether they are Lebesgue measurable is independent from ZFC.

7.C. **Isomorphism of continuous measures.** Let $(X,\mathcal{S}),(Y,\mathcal{P})$ two measurable spaces and $f : X \to Y$ a measurable map. If μ is a measure on $(X,\mathcal{S})$ then we write $f\mu$ for the *push-forward measure* on $(Y,\mathcal{P})$ defined by

$$f\mu(B) = \mu\left(f^{-1}(B)\right).$$

The following theorem is the analogue of the Borel Isomorphism Theorem for measures, and it says that there is only one, up to isomorphism, continuous Borel measure.

The Measure Isomorphism Theorem 7.7. *Let X be a standard Borel space and let μ be a continuous Borel probability measure on X. Then the measure space (X,μ) is Borel isomorphic to $([0,1],\lambda)$, where λ is the Lebesgue measure on $[0,1]$; more precisely, there is a Borel isomorphism $f : X \to [0,1]$ such that the push-forward measure $f\mu$ is equal to λ.*

Proof. By the Borel Isomorphism Theorem, we can assume without loss of generality that $X = [0,1]$. Consider the function

$$g : [0,1] \to [0,1], \text{ by } g(x) = \mu([0,x]).$$

It is immediate that g is increasing and continuous. Moreover, since $g(0) = 0$ and $g(1) = 1$ it is also surjective. We claim that $g\mu = \lambda$. To this end, let $y \in [0,1]$ and pick $x \in [0,1]$ so that $\mu([0,x]) = y$. We have that

$$g\mu([0,y]) = \mu\left(g^{-1}([0,y])\right) = \mu([0,x]) = y.$$

Since $g\mu$ and λ agree on a family which generates the Borel sets, they must agree everywhere. However, g may fail to be an injection. We proceed as follows: for every $y \in [0,1]$, let F_y be the closed interval $g^{-1}(y)$. Let also

$$N = \{y \in [0,1] : F_y \text{ is non-degenerate}\},$$

and notice that N is countable since every non-degenerate interval contains a rational. Let also $M = \bigcup_{y\in N} F_y = g^{-1}(N)$ and notice that $g\restriction_{[0,1]\setminus M}$ is a homeomorphism between $[0,1]\setminus M$ and $[0,1]\setminus N$.

Let $Q \subset [0,1]\setminus N$ be any uncountable Borel set with $\lambda(Q) = 0$ and let $P = g^{-1}(Q)$, so $\mu(P) = 0$. Fix a Borel isomorphism $h : P\cup M \to Q\cup N$ and define the desired $f : [0,1] \to [0,1]$ as follows

$$f(x) = \begin{cases} h(x) & \text{if } x \in P\cup M \\ g(x) & \text{otherwise} \end{cases}.$$

□

7.D. **Abstract Boolean σ-algebras.** Recall that for an a set X, $\mathcal{A} \subseteq \mathscr{P}(X)$ is called an *algebra* if it contains $\varnothing$ and is closed under complements and finite unions. It is called a *σ-algebra* if it is actually closed under countable unions.

Examples 7.8.

(a) In a topological space, clopen sets form an algebra, which in general is not a σ-algebra.

(b) In a measure space, measurable sets form a σ-algebra.

Also recall, that $\mathcal{I} \subseteq \mathscr{P}(X)$ is called an *ideal* if it contains $\varnothing$, is closed downward under $\subseteq$ and is also closed under finite unions. It is called a *σ-ideal* if it is actually closed under countable unions. Notions of "smallness" of sets most often correspond to ideals or even σ-ideals.

Examples 7.9.

(a) In a topological space, nowhere dense sets form an ideal, which in general is not a σ-ideal.

(b) In a measure space, null sets form a σ-ideal.

Given an algebra $\mathcal{A}$ and an ideal $\mathcal{I} \subseteq \mathcal{A}$, we define an equivalence relation $=_{\mathcal{I}}$ on $\mathcal{A}$ by declaring, for $A, B \in \mathcal{A}$,

$$A =_{\mathcal{I}} B \Leftrightarrow A \triangle B \in \mathcal{I}.$$

For $A \in \mathcal{A}$, we let $[A]_{\mathcal{I}}$ denote the $=_{\mathcal{I}}$-equivalence class of A and let $\mathcal{A}/\mathcal{I}$ denote the set of all equivalence classes. Now technically, the quotient set $\mathcal{A}/\mathcal{I}$ is no longer an algebra of subsets of a set, but it is very much like an algebra, in the sense that $[0]_{\mathcal{I}}$ behaves like $\varnothing$, and the operations $\vee$ and $-$ defined by

$$[A]_{\mathcal{I}} \vee [B]_{\mathcal{I}} \coloneqq [A \cup B]_{\mathcal{I}}, \quad -[A]_{\mathcal{I}} \coloneqq [A^c]_{\mathcal{I}},$$

behave like union and complement, respectively. This motivates defining an *abstract Boolean algebra* as a structure $\boldsymbol{A} = (A, 0, \vee, -)$, where A is a set called the domain of $\boldsymbol{A}$, $0 \in A$, $\vee$ is a binary operation called *join* and $-$ is a unary operation called *complement*, and they satisfy certain axioms that make this structure very similar to an actual algebra, with 0 corresponding to $\varnothing$, $\vee$ to union and $-$ to set-complement. For the precise statement of the axioms, we refer to [Hal93]. Using $0, \vee, -$, one can define further operations and a relation as follows: for $a, b \in A$,

$$a \wedge b \coloneqq -\big((-a) \vee (-b)\big), \quad a - b \coloneqq a \wedge (-b), \quad a \leq b :\Leftrightarrow a - b = 0,$$

where we refer to $\wedge$ as *meet*. The axioms of Boolean algebras ensure that $\leq$ is a partial order and intuitively it corresponds to $\subseteq$.

The axioms of Boolean algebras demand that *vee* is associative, which allows applications of $\vee$ to finite subsets of $\{a_n\}_{n<k} \subseteq A$, that is: $\bigvee_{n<k} a_n \coloneqq a_0 \vee (a_1 \vee (\dots)\dots))$. Extending this to countable subsets, one can define an *abstract Boolean σ-algebra* by allowing applications of $\vee$ to countable subsets, i.e. having an operation $\bigvee_{n\in\mathbb{N}}$ called *countable join* that applies to a countable collection of elements of A. We again refer to [Hal93] for the precise definition and the theory of abstract Boolean σ-algebras.

7.E. **The measure algebra** MALG_μ**.** Let X be a standard Borel space and μ a Borel measure on X. Recall that $\mathrm{MEAS}_\mu(X)$ is a σ-algebra and NULL_μ is a σ-ideal, so we have the equivalence relation $=_{\mathrm{NULL}_\mu}$, which we will denote by $=_\mu$ for short. We will also denote by $[A]_\mu$ (or simply $[A]$) the $=_\mu$-equivalence class of $A \in \mathrm{MEAS}_\mu(X)$. We call the Boolean σ-algebra

$$\mathrm{MALG}_\mu(X) := \mathrm{MEAS}_\mu(X)/\mathrm{NULL}_\mu(X),$$

the *measure algebra of* μ. For $[A],[B] \in \mathrm{MALG}_\mu$, we define $d_\mu([A],[B]) := \mu(A \triangle B)$.

Theorem 7.10. *For any Borel probability measure μ on X, the map $d_\mu : \mathrm{MALG}_\mu(X) \to [0,1]$ is a complete separable metric on $\mathrm{MALG}_\mu(X)$.*

Proof. To see that d_μ is indeed a metric notice that $\mu(A \triangle B) = \int |\mathbb{1}_A - \mathbb{1}_B| d\mu$ and that $L^1(\mu)$ is a metric space.

To prove completeness and separability, we first assume without the loss of generality, using Theorem 7.7, that (X,μ) is a disjoint union of $([0,1],\lambda)$ and $(K,\bar{\delta})$, λ is the Lebesgue measure, K is a countable (possibly finite or empty) set, and $\bar{\delta} = \Sigma_{k \in K} c_k \delta_k$ with $c_k > 0$ for every $k \in K$.

Since a sequence $([A_n])_n$ in $\mathrm{MALG}_\mu(X)$ is Cauchy if and only if both $\{[A_n \cap [0,1]]\}_n$ and $\{[A_n \cap K]\}_n$ are Cauchy we can treat each case separately. Restricting to the countable part (assuming $X = K$) notice that for every $k \in X$ since $\bar{\delta}(\{k\}) = c_k > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n > n_0$ $k \in A_n$ if and only if $k \in A_{n_0}$. From this observation we can define(and check that it indeed is) the limit of $\{[A_n]\}_n$. For the continuous case, that is, $(X,\mu) = ([0,1],\lambda)$, one checks that $[A] := [\bigcap_n \bigcup_{i>n} A_i]$ is indeed the limit of $[A_n]$.

Separability for $\mathrm{MALG}_\mu(X)$ is obvious when $X = K$ and for the case $X = [0,1]$ notice that the set of all $[F]$, where F is a finite union of intervals with rational endpoints, is a countable dense subset of $\mathrm{MALG}_\mu(X)$. $\square$

Corollary 7.11. *The space* $(\mathrm{MALG}_\mu(X), d_\mu)$ *is Polish whenever X is Polish and μ is a probability measure on X.*

7.F. **Point realizations of σ-homomorphisms.** A map $\Phi : A \to B$ between the domains of the Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is a *homomorphism* if it respects $\vee, -$ and maps 0_A to 0_B. If moreover Φ is bijective (and hence Φ^{-1} exists and is also a homomorphism), we say that Φ is an *isomorphism*. An *automorphism* of $\boldsymbol{A}$ is an isomorphism between $\boldsymbol{A}$ and $\boldsymbol{A}$. If $\boldsymbol{A}, \boldsymbol{B}$ are Boolean σ-algebras and Φ moreover respects the countable joins $\bigvee_{n \in \mathbb{N}}$ we call Φ a *σ-homomorphism*. Similarly we define *σ-isomorphism* and *σ-automorphism*.

Examples 7.12.

(a) For concrete (σ-)algebras $\mathcal{A} \subseteq \mathscr{P}(X)$, $\mathcal{B} \subseteq \mathscr{P}(Y)$ of subsets, one way to give rise to a (σ-)homomorphism $\Phi : \mathcal{B} to \mathcal{A}$ is via a measurable point-map $\varphi : (X,\mathcal{A}) \to (Y,\mathcal{B})$, which allows for defining $\Phi(B) := \varphi^{-1}(B)$, for $B \in \mathcal{B}$.

(b) Moreover, if $\mathcal{I} \subseteq \mathcal{A}$ is a (σ-)ideal, then

$$\Phi' : \mathcal{B} \to \mathcal{A}/\mathcal{I} \text{ given by } \Phi'(B) := [\varphi^{-1}(B)]_{\mathcal{I}}$$

is still a (σ-)homomorphism;

(c) Furthermore, if $\mathcal{J} \subseteq \mathcal{B}$ is a (σ-)ideal and ϕ has the property that $\varphi^{-1}(\mathcal{J}) \subseteq \mathcal{I}$, then

$$\Phi'' : \mathcal{B}/\mathcal{J} \to \mathcal{A}/\mathcal{I} \text{ given by } \Phi''([B]_{\mathcal{J}}) := [\varphi^{-1}(B)]_{\mathcal{I}}$$

is well-defined and still a (σ-)homomorphism.

(d) As an instance of the latter, letting (X, μ) be a standard Borel space with a Borel measure μ, any Borel automorphism T of X that is μ-nonsingular, i.e. $T^{-1}(\mathrm{NULL}_\mu) \subseteq \mathrm{NULL}_\mu$, gives rise to an automorphism of MALG_μ by $[A] \mapsto [T^{-1}(A)]$.

In all of the above examples, σ-homomorphisms arise from *point-realizations*, i.e. measurable maps from X to Y. The following theorem implies that all σ-homomorphisms of the form $\mathcal{B} \to \mathcal{A}/\mathcal{I}$ admit such point realizations.

Theorem 7.13 (Sikorski)**.** *Let $(X, \mathcal{S})$ be a measurable space, $\mathcal{I} \subseteq \mathcal{S}$ a σ-ideal, Y a non-empty standard Borel space, and $\Phi : \mathcal{B}(Y) \to \mathcal{S}/\mathcal{I}$ a σ-homomorphism. Then, there is a measurable map*

$$\varphi : (X, \mathcal{S}) \to (Y, \mathcal{B}(Y)), \text{ such that } \Phi(B) = [\varphi^{-1}(A)]_{\mathcal{I}}, \text{ for every } B \in \mathcal{B}(Y).$$

Moreover, such φ is unique up to $\mathcal{I}$, i.e. for any other $\psi : (X, \mathcal{S}) \to (Y, \mathcal{B}(Y))$ with $\Phi(B) = [\psi^{-1}(B)]_{\mathcal{I}}$, we have that $\{x \in X : \varphi(x) \neq \psi(x)\} \in \mathcal{I}$.

Proof. By the Borel isomorphism theorem, we assume, without loss of generality, that $Y = [0, 1]$.

For every $p \in [0, 1]$ pick $B_p \in \mathcal{S}$ so that $\Phi([0, p]) = [B_p]_{\mathcal{I}}$, where we particularly put $B_1 := X$ (which we may since $\Phi([0, 1]) = [X]_{\mathcal{I}}$). A necessary condition that a potential $\varphi : X \to Y$ should satisfy is the following: for each $x \in X$,

$$\varphi(x) \in \bigcap_{\substack{p \in [0,1] \\ x \in B_p}} [0, p].$$

This leads us to defining $\varphi(x) := \inf \{p \in [0, 1] \cap \mathbb{Q} : x \in B_p\}$. To see that φ is measurable notice that

$$\varphi^{-1}([0, a]) = \bigcap_{\substack{p < a \\ p \in \mathbb{Q}}} B_p.$$

Now let $\Phi' : \mathcal{B}(Y) \to \mathcal{S}/\mathcal{I}$ with $\Phi'(B) := [\varphi^{-1}(B)]_{\mathcal{I}}$. To see that $\Phi' = \Phi$ notice that

$$\Phi([0, p]) = \Phi\Big(\bigcup_{\substack{q < p \\ q \in \mathbb{Q}}} [0, q]\Big) = \Big[\bigcup_{\substack{q < p \\ q \in \mathbb{Q}}} \Phi([0, q])\Big]_{\mathcal{I}} = \Big[\bigcup_{\substack{q < p \\ q \in \mathbb{Q}}} B_q\Big]_{\mathcal{I}} = \Phi'([0, p]).$$

Since Φ and Φ' agree on intervals of the form $[0, p]$ with $p \in \mathbb{Q}$, they must be equal on all of $\mathcal{B}([0, 1])$ because both are σ-homomorphisms and the σ-algebra generated by such intervals is exactly $\mathcal{B}([0, 1])$.

To see that the map φ is unique let ψ another such map and assume towards a contradiction that $\{x \in X : \varphi(x) < \psi(x)\} \notin \mathcal{I}$ (and without loss of generality, we picked the following direction of the inequality). But then,

$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} \{x \in X : \varphi(x) \leq q < \psi(x)\} \notin \mathcal{I},$$

so for some $q \in \mathbb{Q} \cap [0, 1]$, $\{x \in X : \varphi(x) \leq q < \psi(x)\} \notin \mathcal{I}$, which contradicts

$$\begin{aligned} [\{x \in X : \varphi(x) \leq q\}]_{\mathcal{I}} &= [\varphi^{-1}([0, q])]_{\mathcal{I}} \\ &= \Phi([0, q]) \\ &= [\psi^{-1}([0, q])]_{\mathcal{I}} = [\{x \in X : \psi(x) \leq q\}]_{\mathcal{I}}. \end{aligned}$$

□

Corollary 7.14. *Let X, Y be standard Borel spaces, and $\mathcal{I} \subseteq \mathcal{B}(X)$, $\mathcal{J} \subseteq \mathcal{B}(Y)$ be σ-ideals. A map $\Phi : \mathcal{B}(X)/\mathcal{I} \to \mathcal{B}(Y)/\mathcal{J}$ is a σ-isomorphism if and only if there is a Borel isomorphism $\varphi : Y' \to X'$, where $Y' =_{\mathcal{J}} Y$ and $X' =_{\mathcal{I}} X$, such that $\Phi([A]_{\mathcal{I}}) = [\phi^{-1}(A)]_{\mathcal{J}}$ for all $A \in \mathcal{B}(X)$. Moreover, if both $\mathcal{I}$ and $\mathcal{J}$ contain uncountable sets, then X' and Y' can be taken to be X and Y, respectively.*

Proof. Let $\varphi : Y \to X$ and $\psi : X \to Y$ be Borel maps such that $\Phi(A) = [\varphi^{-1}(A)]_{\mathcal{J}}$ and $\Phi^{-1}(B) = [\psi^{-1}(B)]_{\mathcal{I}}$. Then $\Phi \circ \Phi^{-1}$ has $\psi \circ \varphi$ as a point realization and similarly $\Phi^{-1} \circ \Phi$ has $\varphi \circ \psi$ as a point realization. So by the uniqueness in Theorem 7.13, we get that $\psi \circ \varphi =_{\mathcal{J}} \mathrm{id}_Y$ and $\varphi \circ \psi =_{\mathcal{I}} \mathrm{id}_X$. □

Taking $X = Y$ and $\mathcal{I} = \mathcal{J} = \mathrm{NULL}_\mu$ for a Borel measure μ on X, this corollary gives that any automorphism of MALG_μ arises from a μ-nonsingular Borel automorphism of (X, μ).

8. Baire category

8.A. **Nowhere dense sets.** Let X be a topological space. A set $A \subseteq X$ is said to be dense in $B \subseteq X$ if $A \cap B$ is dense in B.

Definition 8.1. Let X be a topological space. A set $A \subseteq X$ is called **nowhere dense** if there is no nonempty open set $U \subseteq X$ in which A is dense.

Proposition 8.2. *Let X be a topological space and $A \subseteq X$. The following are equivalent:*

(1) A is nowhere dense;
(2) A misses a nonempty open subset of every nonempty open set (i.e. for every open set $U \neq \varnothing$ there is a nonempty open subset $V \subseteq U$ such that $A \cap V = \varnothing$);
(3) The closure $\overline{A}$ has empty interior.

Proof. Follows from definitions. □

Proposition 8.3. *Let X be a topological space and $A, U \subseteq X$.*

(a) A is nowhere dense if and only if $\overline{A}$ is nowhere dense.
(b) If U is open, then $\partial U := \overline{U} \setminus U$ is closed nowhere dense.
(c) If U is open dense, then U^c is closed nowhere dense.
(d) Nowhere dense subsets of X form an ideal[4].

Proof. Part (a) immediately follows from (2) of Proposition 8.2. For (b) note that ∂U is disjoint from U so its interior cannot be nonempty. Since it is also closed, it is nowhere dense by (2) of Proposition 8.2, again. As for part (c), it follows directly from (b) because by the density of U, $\partial U = U^c$. Finally, we leave part (d) as an easy exercise. □

For example, the Cantor set is nowhere dense in $[0, 1]$ because it is closed and has empty interior. Also, any compact set K is nowhere dense in $\mathcal{N}$ because it is closed and the corresponding tree $T_K \subseteq \mathbb{N}^{<\mathbb{N}}$ is finitely branching.

[4]An *ideal* on a set X is a collection of subsets of X containing $\varnothing$ and closed under subsets and finite unions.

8.B. **Meager sets.**

Definition 8.4. Let X be a topological space. A set $A \subseteq X$ is **meager** if it is a countable union of nowhere dense sets. The complement of a meager set is called **comeager**.

Note that the family $\mathrm{MGR}(X)$ of meager subsets of X is a σ-ideal[5] on X; in fact, it is precisely the σ-ideal generated by nowhere dense sets. Consequently, comeager sets form a countably closed filter[6] on X.

An example of a meager set is any σ-compact set is in $\mathcal{N}$. Also, any countable set in a nonempty perfect space is meager, so, for example, $\mathbb{Q}$ is meager in $\mathbb{R}$.

Meager sets often have properties analogous to those enjoyed by the null sets in R^n (with respect to the Lebesgue measure). The following proposition lists some of them.

Proposition 8.5. *Let X be a topological space and $A \subseteq X$.*

(a) A is meager if and only if it is contained in a countable union of closed nowhere dense sets. In particular, every meager set is contained in a meager F_σ set.

(b) A is comeager if and only if it contains a countable intersection of open dense sets. In particular, dense G_δ sets are comeager.

Proof. Part (b) follows from (a) by taking complements, and part (a) follows directly from the corresponding property of nowhere dense sets proved above. ☐

Remark 8.6. Part (b) of the above proposition provides a method of proving that a given set is comeager by showing that it is dense G_δ or that it contains a dense G_δ subset.

As an application of some of the statements above, we record the following random fact.

Proposition 8.7. *Every second countable space X contains a dense G_δ (hence comeager) subset Y that is zero-dimensional in the relative topology.*

Proof. Indeed, if $\{U_n\}_{n\in\mathbb{N}}$ is a basis for X, then $F = \bigcup_n (\overline{U}_n \setminus U_n)$ is meager F_σ and $Y = X \setminus F$ is zero-dimensional. ☐

8.C. **Relativization of nowhere dense and meager.**

Let X be a topological space and P be a property of subsets of X (e.g. open, closed, compact, nowhere dense, meager). We say that property P is *absolute between subspaces* if for every subspace $Y \subseteq X$ and $A \subseteq Y$, A has property P as a subset of Y iff it has property P as a subset of X. An example of a property that is absolute between subspaces is compactness (why?), but I can't think of any other absolute property. It is clear that properties like open or closed are not absolute. Furthermore, nowhere dense is not absolute: let $X = \mathbb{R}$ and $A = Y = \{0\}$. Now A is clearly nowhere dense in $\mathbb{R}$ but in Y all of a sudden it is, in fact, open, and hence not nowhere dense. Thus being nowhere dense does not *transfer downward* (from a bigger space to a smaller subspace); same goes for meager. However, the following proposition shows that it *transfers upward* and that it is absolute between open subspaces.

Proposition 8.8. *Let X be a topological space, $Y \subseteq X$ be a subspace and $A \subseteq Y$.*

(a) If A is nowhere dense (resp. meager) in Y, it is still nowhere dense (resp. meager) in X.

[5] An *σ-ideal* on a set X is an ideal that is closed under countable unions.

[6] A *filter* on a set X is the dual to an ideal on X, more precisely, it is a collection of subsets of X containing X and closed under supersets and finite intersections. If moreover, it is closed under countable intersections, we say that it is *countably closed*.

(b) If Y is open, then A is nowhere dense (resp. meager) in Y iff it is nowhere dense (resp. meager) in X.

Proof. Straightforward, using (2) of Proposition 8.2. □

8.D. **Baire spaces.** Being a σ-ideal is a characteristic property of many notions of "smallness" of sets, such as being countable, having measure 0, etc, and meager is one of them. However, it is possible that a topological space X is such that X itself is meager, so the σ-ideal of meager sets trivializes, i.e. is equal to $\mathscr{P}(X)$. The following definition isolates a class of spaces where this doesn't happen.

Definition 8.9. A topological space is said to be Baire if every nonempty open set is non-meager.

Proposition 8.10. *Let X be a topological space. The following are equivalent:*
(1) X is a Baire space, i.e. every nonempty open set is non-meager.
(2) Every comeager set is dense.
(3) The intersection of countably many dense open sets is dense.

Proof. For (1) $\Leftrightarrow$ (2), note that a nonempty open set is nonmeager if and only if every comeager set meets it. The equivalence (2) $\Leftrightarrow$ (3) follows directly from (b) of Proposition 8.5. □

As mentioned above, in any topological space, dense G_δ sets are comeager. Moreover, by the last proposition, we have that in Baire spaces any comeager set contains a dense G_δ set. So we get:

Corollary 8.11. *In Baire spaces, a set is comeager if and only if it contains a dense G_δ set.*

Proposition 8.12. *If X is a Baire space and $U \subseteq X$ is open, then U is a Baire space.*

Proof. Follows from (b) of Proposition 8.8. □

Baire Category Theorem 8.13. *Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.*

Proof. We will only prove for completely metrizable spaces and leave the locally compact Hausdorff case as an exercise. Let (X, d) be a complete metric space and let $(U_n)_{n\in\mathbb{N}}$ be dense open. Let U be nonempty open and we show that $\bigcap_n U_n \cap U \neq \varnothing$. Put $V_0 = U$ and since $U_0 \cap V_0 \neq \varnothing$, there is a nonempty open set V_1 of diameter < 1 such that $\overline{V}_1 \subseteq U_0 \cap V_0$. Similarly, since $U_1 \cap V_1 \neq \varnothing$, there is a nonempty open set V_2 of diameter $< 1/2$ such that $\overline{V}_2 \subseteq U_1 \cap V_1$, etc. Thus there is a decreasing sequence $(\overline{V}_n)_{n\geq 1}$ of nonempty closed sets with vanishing diameter ($\operatorname{diam}(V_n) < 1/n$) and such that $\overline{V}_n \subseteq U_n \cap U$. By the completeness of X, $\bigcap_n \overline{V}_n$ is nonempty (is, in fact, a singleton) and hence so is $\bigcap_n U_n \cap U$. □

Thus, Polish spaces are Baire and hence comeager sets in them are "truly large", i.e. they are not meager! This immediately gives:

Corollary 8.14. *In Polish spaces, dense meager sets are not G_δ. In particular, $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$.*

Proof. If a subset is dense G_δ, then it is comeager, and hence nonmeager. □

Definition 8.15. Let X be a topological space and $P \subseteq X$. If P is comeager, we say that P holds generically or that the generic element of X is in P. (Sometimes the word typical is used instead of generic.)

In a nonempty Baire space X, if $P \subseteq X$ holds generically, then, in particular, $P \neq \varnothing$. This leads to a well-known method of existence proofs in mathematics: in order to show that a given set $P \subseteq X$ is nonempty, where X is a nonempty Baire space, it is enough to show that P holds generically. Although the latter task seems harder, the proofs are often simpler since having a notion of largeness (like non-meager, uncountable, positive measure) allows using pigeon hole principles and counting, whereas constructing a concrete object in P is often complicated. The first example of this phenomenon was due to Cantor who proved the existence of transcendental numbers by showing that there are only countably many algebraic ones, whereas reals are uncountable, and hence, "most" real numbers are transcendental. Although the existence of transcendental numbers was proved by Liouville before Cantor, the simplicity of Cantor's proof and the apparent power of the idea of counting successfully "sold" Set Theory to the mathematical community.

8.E. **The Baire property.** Let $\mathcal{I}$ be a σ-ideal on a set X. For $A, B \subseteq X$, we say that A and B are equal modulo $\mathcal{I}$, noted $A =_{\mathcal{I}} B$, if the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A) \in \mathcal{I}$. This is clearly an equivalence relation that respects complementation and countable unions/intersections.

In the particular case where $\mathcal{I}$ is the σ-ideal of meager sets of a topological space, we write $A =^* B$ if A, B are equal modulo a meager set.

Definition 8.16. Let X be a topological space. A set $A \subseteq X$ has the Baire property (BP) if $A =^* U$ for some open set $U \subseteq X$.

For a topological space X, let $\mathrm{BP}(X)$ denote the collection of all subsets of X with the BP.

Proposition 8.17. $\mathrm{BP}(X)$ *is a σ-algebra on X. In fact, it is the smallest σ-algebra containing all open sets and all meager sets.*

Proof. The second assertion follows from the first and the fact that any set $A \in \mathrm{BP}(X)$ can be written as $A = U \Delta M$, where U is open and M is meager.

For the first assertion, we start by noting that if U is open, then $\overline{U} \setminus U$ is closed and nowhere dense, so $U =^* \overline{U}$. Taking complements, we see that if F is closed, $F \setminus \mathrm{Int}(F)$ is closed nowhere dense, so $F =^* \mathrm{Int}(F)$, and hence closed sets have the BP. This implies that BP is closed under complements because if A has the BP, then $A =^* U$ for some open U, and thus $A^c =^* U^c =^* \mathrm{Int}(U^c)$, so A^c has the BP. Finally, if each A_n has the BP, say $A_n =^* U_n$ with U_n open, then $\bigcup_n A_n =^* \bigcup_n U_n$, so $\bigcup_n A_n$ has the BP. □

In particular, all open, closed, F_σ, G_δ, and in general, all Borel sets, have the BP. How far can we push this? The following is a non-trivial theorem, whose proof we omit here.

Theorem 8.18. *Analytic (and hence also co-analytic) subsets of Polish spaces have the* BP.

This theorem might make one hopeful that perhaps all projective sets have the BP. However, it turns out that already for $\mathbf{\Sigma}^1_2$ sets whether or not the BP holds is independent from ZFC.

Proposition 8.19. *Let X be a topological space and $A \subseteq X$. Then the following are equivalent:*

(1) A has the BP*;*
(2) $A = G \cup M$, where G is G_δ and M is meager;
(3) $A = F \setminus M$, where F is F_σ and M is meager.

Proof. Follows the fact every meager set is contained in a meager F_σ set (see Proposition 8.5). □

Definition 8.20. For topological spaces X, Y, a function $f : X \to Y$ is called Baire measurable if the preimage of every open set has the BP.

Proposition 8.21. *Let X, Y be topological space and suppose Y is second countable. Then any Baire measurable function $f : X \to Y$ is continuous on a comeager set, i.e. there is a comeager set $D \subseteq X$ such that $f\restriction_D : D \to Y$ is continuous.*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis for Y. Because f is Baire measurable, $f^{-1}(V_n) =^* U_n$ for some open set $U_n \subseteq X$. Put $M_n = f^{-1}(V_n)\Delta U_n$ and let $D = X \setminus \bigcup_n M_n$. Now to show that $f\restriction_D$ is continuous, it is enough to check that for each n, $(f\restriction_D)^{-1}(V_n) = U_n \cap D$. For this, just note that $(f\restriction_D)^{-1}(V_n) = f^{-1}(V_n) \cap D$, and since $M_n \cap D = \varnothing$, we have $f^{-1}(V_n) \cap D = U_n \cap D$. □

8.F. **Localization.** Recall that nonempty open subsets of Baire spaces are Baire themselves in the relative topology and all of the notions of category are absolute when relativizing to an open subset. This allows us to localize the notions of category to open sets.

Definition 8.22. Let X be a topological space and $U \subseteq X$ an open set. We say that A is meager in U if $A \cap U$ is meager in X[7] and A is comeager in U if $U \setminus A$ is meager. If A is comeager in U, we say that A *holds generically in* U or that U *is a $*$-subset of* A, in symbols $U \subseteq^* A$[8].

Thus, A is comeager iff $X \subseteq^* A$. We now have the following simple fact that will be used over and over in our arguments below.

Proposition 8.23 (Baire alternative)**.** *Let A be a set with the* BP *in a topological space X. Then either A is meager or $U \subseteq^* A$ for some nonempty open $U \subseteq X$[9]. If X is Baire, then exactly one of these alternatives holds.*

Proof. By the BP, $A =^* U$ for some open U. If $U = \varnothing$, then A is meager; otherwise, $U \neq \varnothing$ and $U \subseteq^* A$. □

8.G. **The Kuratowski–Ulam theorem.** In this subsection we state without proof an analog of Fubini's theorem for Baire category. We start by fixing a convenient notation.

Let X be a topological space. For a set $A \subseteq X$ and $x \in X$, we put

$$A(x) \iff x \in A,$$

[7]This is equivalent to $A \cap U$ being meager relative to U.
[8]The more set-theoretic terminology is that U *forces* A and it is denoted by $U \Vdash A$.
[9]Both alternatives can hold if the space X is not Baire.

viewing A as a property of elements of X and writing $A(x)$ to mean that x has this property. We also use the following notation:

$$\begin{aligned} \forall^* x A(x) &\iff A \text{ is comeager,} \\ \exists^* x A(x) &\iff A \text{ is non-meager.} \end{aligned}$$

We read $\forall^*$ as "for comeager many" x, and $\exists^*$ as "for non-meager many" x.

Similarly, for $U \subseteq X$ open, we write

$$\begin{aligned} \forall^* x \in U A(x) &\iff A \text{ is comeager in } U, \\ \exists^* x \in U A(x) &\iff A \text{ is non-meager in } U. \end{aligned}$$

Thus, denoting the negation by $\neg$, we have:

$$\neg \forall^* x \in U A(x) \iff \exists^* x \in U A^c(x).$$

Recall that for arbitrary topological spaces $X \times Y$, the projection function $\mathrm{proj}_X : X \times Y \to X$ defined by $(x, y) \mapsto x$ is continuous and open (images of open sets are open). Conversely, for every $y \in Y$, the function $X \to X \times Y$ defined by $x \mapsto (x, y)$ is also continuous and open, and hence an embedding.

Theorem 8.24 (Kuratowski–Ulam)**.** *Let X, Y be second countable topological spaces. Let $A \subseteq X \times Y$ have the* BP*, and denote $A_x = \{y \in Y : A(x, y)\}$, $A^y = \{x \in X : A(x, y)\}$.*

(i) $\forall^* x(A_x$ has the BP *in $Y)$. Similarly, $\forall^* y(A_y$ has the* BP *in $X)$.*
(ii) A is meager $\iff \forall^* x(A_x$ is meager$) \iff \forall^* y(A_y$ is meager$)$.
(iii) A is comeager $\iff \forall^* x(A_x$ is comeager$) \iff \forall^* y(A_y$ is comeager$)$. In symbols:

$$\forall^*(x, y) A(x, y) \iff \forall^* x \forall^* y A(x, y) \iff \forall^* y \forall^* x A(x, y).$$

The Kuratowski–Ulam theorem fails if A does not have the BP. For example, using AC, one can construct a non-meager set $A \subseteq \mathbb{R}^2$ so that no three points of A are on a straight line.

8.H. **Generically ergodic group actions.** Let X be a topological space and let G be a group acting on X by homeomorphisms, i.e. each $g \in G$ is a homeomorphism of X. A subset $A \subseteq X$ is called invariant if it is closed under the action, i.e. $G \cdot A = A$; equivalently, A is a union of orbits. For a set $A \subseteq X$, the *saturation of* A, noted $[A]_G$, is the smallest invariant subset containing A; equivalently, $[A]_G \coloneqq G \cdot A = \bigcup_{g \in G} gA$.

Definition 8.25. The action $G \curvearrowright X$ is called *generically ergodic* if every invariant subset with the BP is either meager or comeager.

In other words, generic ergodicity means that the action is topologically irreducible/atomic.

Theorem 8.26 (The first topological $0-1$ law)**.** *Let X be a Baire space and let G be a group acting on X by homeomorphisms. The following are equivalent:*

(1) *$G \curvearrowright X$ is generically ergodic.*
(2) *Every invariant nonempty open set is dense.*
(3) *(Homogeneity) For any nonempty open sets $U, V \subseteq X$, there is $g \in G$ such that $g(U) \cap V \neq \varnothing$.*

If X is second countable, then we have two more equivalent conditions:

(4) *For comeager-many $x \in X$, the orbit of x is dense.*
(5) *There is a dense orbit.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.
(3) $\Rightarrow$ (1): Let $A \subseteq X$ be an invariant set with the BP and suppose towards a contradiction that both A, A^c are non-meager. Then, by the Baire alternative, there are nonempty open sets $U, V \subseteq X$ with $U \subseteq^* A$ and $V \subseteq^* B$. Taking $g \in G$ with $W := gU \cap V \neq \varnothing$, we see that $W \subseteq^* A$ and $W \subseteq^* A^c$, so $W \subseteq^* \varnothing$, which means that W is meager, contradiction X being Baire.
(2) $\Rightarrow$ (4): Let $(U_n)_n$ be a countable basis for X. The saturation $[U_n]_G$ of each U_n is open because $[U_n]_G = \bigcup_{g \in G} gU_n$, so by (2), $[U_n]_G$ is dense. Therefore, $X' := \bigcap_n [U_n]_G$ is comeager being dense G_δ; moreover, X' is invariant. Now for any $x \in X$, and any $n \in \mathbb{N}$,

$$[x]_G \cap U_n \neq \varnothing \iff x \in [U_n]_G,$$

so if $x \in X'$, then the orbit $[x]_G$ meets every basic open set U_n.
(4) $\Rightarrow$ (5): Follows because X is Baire.
(5) $\Rightarrow$ (3): Straightforward from the definitions. $\square$

Part 4. Appendix on equivalence relations and Polish group actions

For the past twenty five years, a major focus of descriptive set theory has been the study of equivalence relations on Polish spaces that are definable when viewed as sets of pairs (e.g. orbit equivalence relations of continuous actions of Polish groups are analytic). This study is motivated by foundational questions such as understanding the nature of classification of mathematical objects (measure-preserving automorphisms, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, conformal equivalence, etc.), and creating a mathematical framework for measuring the complexity of such classification problems. Due to its broad scope, it has natural interactions with other areas of mathematics, such as ergodic theory and topological dynamics, functional analysis and operator algebras, representation theory, topology, model theory, etc.

The following definition makes precise what it means for one classification problem to be easier (not harder) than another.

Definition. Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that E is Borel reducible to F and write $E \leq_B F$ if there is a Borel map $f : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1 \iff f(x_0) F f(x_1)$.

We call E *smooth* (or *concretely classifiable*) if it Borel reduces to the identity relation $\mathrm{id}(X)$ on some (any) Polish space X (note that such E is automatically Borel). An example of such an equivalence relation is the similarity relation of matrices; indeed, if $J(A)$ denotes the Jordan canonical form of a matrix $A \in \mathbb{R}^{n^2}$, then for $A, B \in \mathbb{R}^{n^2}$, we have $A \sim B \iff J(A) = J(B)$. It is not hard to check that the computation of $J(A)$ is Borel, so $J : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is a Borel reduction of $\sim$ to $\mathrm{id}(\mathbb{R}^{n^2})$, and hence $\sim$ is smooth. Another (much more involved) example is the isomorphism of Bernoulli shifts, which, by Ornstein's famous theorem, is reduced to the equality on $\mathbb{R}$ by the map assigning to each Bernoulli shift its entropy.

However, many equivalence relations that appear in mathematics are nonsmooth. For example, the Vitali equivalence relation $\mathbb{E}_v$ on $[0,1]$ defined by $x \mathbb{E}_v y \iff x - y \in \mathbb{Q}$ can be easily shown to be nonsmooth using measure-theoretic or Baire category arguments. The following theorem (known as the General Glimm–Effros dichotomy, see [HKL90]) shows that in fact containing $\mathbb{E}_v$ is the only obstruction to smoothness:

Theorem (Harrington–Kechris–Louveau '90)**.** *Let E be a Borel equivalence relation on a Polish space X. Then either E is smooth, or else $\mathbb{E}_v \sqsubseteq_B E$.*[10]

This was one of the first major victories of descriptive set theory in the study of equivalence relations. It in particular implies that $\mathbb{E}_v$ is the easiest among all nonsmooth Borel equivalence relations in the sense of Borel reducibility. Besides its foundational importance in the theory of Borel equivalence relations, it also generalized earlier important results of Glimm and Effros. By now, many other dichotomy theorems have been proved and general

[10]Here, $\sqsubseteq_B$ means that there is an injective Borel reduction.

methods of placing a given equivalence relation among others in the Borel reducibility hierarchy have been developed. However, there are still many fascinating open problems left and the Borel reducibility hierarchy is yet to be explored.

9. Examples of equivalence relations and Polish group actions

9.A. **Equivalence relations.** Let X denote a Polish space. We start by listing some familiar examples of equivalence relations that appear in various areas of mathematics.

Examples 9.1.

(a) The identity (equality) relation $\mathrm{Id}(X)$ on X is a closed equivalence relation.

(b) The Vitali equivalence relation $\mathbb{E}_v$ on $[0,1]$, defined by $x\mathbb{E}_v y \iff x - y \in \mathbb{Q}$, is clearly an F_σ equivalence relation.

(c) Define the equivalence relation $\mathbb{E}_0(X)$ on $X^{\mathbb{N}}$ of eventual equality of sequences, namely: for $x, y \in X^{\mathbb{N}}$, $x\mathbb{E}_0(X)y \Leftrightarrow \forall^\infty n(x(n) = y(n))$. This is again an F_σ equivalence relation. Important special cases when $X = 2$, i.e. $X^{\mathbb{N}} = \mathcal{C}$, and when $X = \mathcal{N}$. In the first case we simply write $\mathbb{E}_0 \coloneqq \mathbb{E}_0(2)$ and in the second case we write $\mathbb{E}_1 \coloneqq \mathbb{E}_0(\mathcal{N})$.

(d) The similarity relation $\sim$ of matrices on the space $M_n(\mathbb{C})$ of $n \times n$ matrices: for $A, B \in M_n(\mathbb{C})$, $A \sim B \Leftrightarrow \exists Q \in GL_n(\mathbb{C})\ QAQ^{-1} = B$. By definition, this is an analytic equivalence relation, but we will see below that it is actually Borel.

(e) Consider the following subgroups of $\mathbb{R}^{\mathbb{N}}$ under addition:
- $\ell_p = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x(n)|^p < \infty\}$, for $1 \le p < \infty$,
- $\ell_\infty = \{x \in \mathbb{R}^{\mathbb{N}} : \sup_n |x(n)| < \infty\}$,
- $c_0 = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_n x(n) = 0\}$.

The first two are F_σ subsets of $\mathbb{R}^{\mathbb{N}}$ and the last is $\mathbf{\Pi}^0_3$. Thus, if $\mathcal{I}$ is one of these subgroups, then the equivalence relation $E_{\mathcal{I}}$ on $\mathbb{R}^{\mathbb{N}}$, defined by

$$xE_{\mathcal{I}}y \Leftrightarrow x - y \in \mathcal{I},$$

is F_σ for $\mathcal{I} = \ell_p$, $1 \le p \le \infty$, and is $\mathbf{\Pi}^0_3$ for $\mathcal{I} = c_0$.

(f) Fix a countable first-order relational language $\mathcal{L} = \{R_i\}_{i\in\mathbb{N}}$, where R_i is a relation symbol of arity n_i. The set of countable $\mathcal{L}$-structures can be turned into a Polish space by fixing their underlying set to be $\mathbb{N}$ and, for each i, identifying the interpretation of R_i (i.e. a relation on N^{n_i}) with its characteristic function. Such a structure is simply an element of $X_{\mathcal{L}} \coloneqq \prod_{i\in\mathbb{N}} 2^{N^{n_i}}$. This allows talking about the Polish spaces of countable orderings and countable graphs, for example. Also, because any first-order language can be turned into a relational language by replacing function symbols with relation symbols for their graphs, we can also consider Polish spaces of countable groups, rings, fields, etc.

Thus, *isomorphism of countable $\mathcal{L}$-structures*, denoted by $\simeq_{\mathcal{L}}$, naturally falls into the framework of descriptive set theory as it is an analytic equivalence relation on $X_{\mathcal{L}}$; indeed, two structures are isomorphic if and only if there exists a certain bijection f from $\mathbb{N}$ to $\mathbb{N}$, i.e. a certain element $f \in \mathcal{N}$.

9.B. **Polish groups.** Many natural analytic equivalence relations arise as orbit equivalence relations of continuous (or Borel) actions of Polish groups.

Definition 9.2. A topological group is a group with a topology on it so that group multiplication $(x, y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Such a group is called Polish if its topology happens to be Polish.

Here are some important examples of Polish groups.

Examples 9.3.

(a) All countable groups with the discrete topology are Polish. In fact, it is an exercise to show that the only Polish topology on a countable group is the discrete topology.

(b) The unit circle $S^1 \subseteq \mathbb{C}$ is a Polish group under multiplication.

(c) $\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ are Polish groups under coordinatewise addition (note that the latter is just the Cantor space $\mathcal{C}$).

(d) The group S_∞ of permutations of $\mathbb{N}$ (i.e. bijections from $\mathbb{N}$ to $\mathbb{N}$) is a G_δ subset of $\mathcal{N}$, so is a Polish group with the relative topology.

(e) The automorphisms group $\mathrm{Aut}(\mu)$ of a standard probability space (X, μ) (our main hero) is a Polish group in the weak topology. See Subsection 1.B for details.

(f) The group $U(\mathcal{H})$ of unitary automorphisms of a separable Hilbert space $\mathcal{H}$ is a Polish group in the strong (equivalently weak) operator topology. See Subsection 1.C for details.

9.C. **Actions of Polish groups.**

Definition 9.4. Let G be a Polish group and X be a Polish space. An action $a : G \curvearrowright X$ of G on X is said to be continuous (resp. Borel) if the action function $a : G \times X \to X$ given by $(g, x) \mapsto g \cdot_a x$ is continuous (resp. Borel).

We denote by E_G (or sometimes by E_a) the orbit equivalence relation induced by such an action. Note that E_G is analytic because for $x, y \in X$,

$$xE_G y \iff \exists g \in G(g \cdot_a x = y).$$

Here we list some examples of continuous actions of Polish groups.

Examples 9.5.

(a) Any Polish group acts on itself by left multiplication, as well as by conjugation. It follows from the definition of topological groups that these actions are continuous.

(b) Let G be a Polish group and $H < G$ be a Polish (equivalently, closed) subgroup. The left multiplication action of H on G is clearly continuous and the induced orbit equivalence relation E_H is the relation of being in the same left H-coset, i.e. $xE_H y \Leftrightarrow Hx = Hy$. We refer to E_H as the *H-coset equivalence relation.*

(c) The Vitali equivalence relation $\mathbb{E}_v$ is exactly the orbit equivalence relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$.

(d) The relation $\mathbb{E}_0$ of eventual equality on $2^{\mathbb{N}}$ is induced by a continuous action of a countable group as follows: for $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t|$, let $\phi_{s,t} : X \to X$ be defined as follows:

$$\phi_{s,t}(x) = \begin{cases} t^\frown y & \text{if } x = s^\frown y \\ s^\frown y & \text{if } x = t^\frown y \\ x & \text{otherwise} \end{cases},$$

and let G be the group generated by $\{\phi_{s,t} : s, t \in \mathbb{N}^{<\mathbb{N}}, |s| = |t|\}$. It is clear that each $\phi_{s,t}$ is a homeomorphism of X and $E_G = \mathbb{E}_0$.

One can also show that after throwing away two orbits (more precisely, restricting $\mathbb{E}_0$ to $X = \{x \in 2^{\mathbb{N}} : x \text{ has infinitely many 0-s and 1-s}\}$), we can realize $\mathbb{E}_0$ by a continuous action of $\mathbb{Z}$. We leave this as an exercise.

(e) Irrational rotation of S^1 is simply an action $\mathbb{Z} \curvearrowright S^1$, where $1 \in \mathbb{Z}$ acts as multiplication by $e^{\alpha\pi i}$, for some irrational $\alpha \in \mathbb{R}$. Clearly this action is continuous and we denote the orbit equivalence relation by $\mathbb{E}_\alpha$.

(f) The similarity relation $\sim$ of matrices in $M_n(\mathbb{R})$ is induced as the orbit equivalence relation of the conjugation action of $GL_n(\mathbb{R})$ on $M_n(\mathbb{R})$.

(g) For a first-order relational language $\mathcal{L}$, the group S_∞ admits a natural action on the Polish space $X_{\mathcal{L}}$ of countable $\mathcal{L}$-structures by permuting their underlying sets. Clearly, the induced orbit equivalence relation is exactly the relation of isomorphism of $\mathcal{L}$-structures.

(h) For a standard probability space (X, μ), one can easily verify that the natural action of $\mathrm{Aut}(X, \mu)$ on (X, μ) is continuous.

(i) Similarly, for a separable Hilbert space $\mathcal{H}$, the natural action of $U(\mathcal{H})$ on $\mathcal{H}$ is continuous.

10. Borel reducibility

Let E and F be equivalence relations on Polish spaces X and Y, respectively. The following defines the class of functions from X to Y that induce functions from X/E to Y/F.

Definition 10.1. A function $f : X \to Y$ is called a homomorphism from E to F if for all $x_0, x_1 \in X$,

$$x_0 E x_1 \Rightarrow f(x_0) F f(x_1).$$

$f : X \to Y$ is called a reduction of E to F if for all $x_0, x_1 \in X$,

$$x_0 E x_1 \Leftrightarrow f(x_0) F f(x_1).$$

Note that reductions induce injections $X/E \hookrightarrow Y/F$.

The following makes it precise what it means for a classification problem in mathematics to be easier (not harder) than another classification problem:

Definition 10.2. Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that E is Borel reducible to F, and write $E \leq_B F$, if there is a Borel reduction of E to F. Furthermore, we say that E is strictly below F, and write $E <_B F$, if $E \leq_B F$ but $F \nleq_B E$.

The choice of "Borel" as the regularity condition on the reduction is mainly because any two uncountable Polish spaces are Borel isomorphic, so the existence of Borel reductions does not depend on the particular choice of the underlying Polish spaces and it only depends on the inherent complexity of the equivalence relations, which is what we want to measure.

We replace the superscript B in $\leq_B$ by c if there is a continuous reduction, and we write $\sqsubseteq$ instead of $\leq$ if the reduction is injective.

It is clear that $\leq_B$ is a quasi-order[11] on the class of all equivalence relations on Polish spaces[12]. We call E and F *Borel bireducible*, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. Since Borel reductions induce Borel embeddings $X/E \hookrightarrow Y/E$, we refer to the bireducibility class of E as *the Borel cardinality of* X/E.

We also call E and F Borel isomorphic, and write $E \simeq_B F$, if there is a bijective Borel reduction (thus a Borel isomorphism from X to Y) of E to F.

Remark 10.3. In general, $E \sqsubseteq_B F$ and $F \sqsubseteq_B E$ does not imply $E \simeq_B F$; more precisely, the Schröder–Bernstein argument doesn't work unless both reductions are "locally surjective on classes". The latter means that the Borel reduction $f : X \to Y$ has the property that for every $x \in X$, $f([x]_E)$ is an entire F-class, i.e. $f([x]_E) = [f(x)]_F$. Indeed, imagine a situation of having injective Borel reductions $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$ of E to F and F to E, respectively, such that for some $x \in X$, $[x]_E \cap g(Y) = \varnothing$ and $f([x]_E) \subsetneq [f(x)]_F$. Then, the Schröder–Bernstein argument would map the elements of $[x]_E$ by f into $[f(x)]_F$ and the elements in $[f(x)]_F \setminus f([x]_E)$ by g into $[g(f(x))]_E$. But $[g(f(x))]_E \neq [x]_E$ because $[x]_E$ is disjoint from $[g(Y)]_F$, so the elements in $f([x]_E)$ would go to a different E-class (namely, $[x]_E$) than the elements in $[f(x)]_F \setminus f([x]_E)$, and hence, the resulting map will not be a reduction.

The systematic study of the Borel reducibility hierarchy of definable equivalence relations is sometimes referred to as *invariant descriptive set theory*. It was pioneered by Silver, Harrington, Kechris, Louveau, and others, in the late '80s and early '90s. The goal of invariant descriptive set thoery is to understand the Borel reducibility hierarchy (and hence, the complexity of classification problems that appear in many areas of mathematics such as analysis, ergodic theory, operator algebras, model theory, recursion theory, etc.), and to develop methods for placing a given equivalence relation into its "correct" spot in this hierarchy.

11. Concrete classifiability (smoothness)

In this section we make it precise what it means to classify mathematical objects (matrices, measure-preserving transformations, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, conformal equivalence, etc.). We will consider some examples and nonexamples, as well as discuss related (famous) dichotomy theorems.

11.A. **Definitions.**

Definition 11.1. An equivalence relation E on a Polish space X is called concretely classifiable (or smooth) if $E \leq_B \mathrm{Id}(\mathbb{R})$. By the Borel isomorphism theorem, $\mathbb{R}$ can be replaced by any other Polish space.

[11] Quasi-order is a reflexive and transitive relation, not necessarily antisymmetric.

[12] This is actually a set if we fix a particular uncountable Polish space, which we can do as any two of them are Borel isomorphic.

Note that *smooth equivalence relations are necessarily Borel*: indeed, if $f : X \to \mathbb{R}$ is a Borel reduction of E to $\mathrm{Id}(\mathbb{R})$, then the function $f_2 : X^2 \to \mathbb{R}^2$ by $(x,y) \mapsto (f(x), f(y))$ is Borel and $E = f_2^{-1}(\Delta_{\mathbb{R}})$, where $\Delta_{\mathbb{R}}$ is the diagonal in $\mathbb{R}^2$. But $\Delta_{\mathbb{R}}$ is closed in $\mathbb{R}^2$, so E is Borel being a preimage of Borel.

A special case of smoothness is when we can select a canonical representative from each equivalence class.

Definition 11.2. Let E be an equivalence relation on a Polish space X. A map $s : X \to X$ is called a selector for E if for all $x \in X$, $s(x) \in [x]_E$, and s is a reduction of E to $\mathrm{Id}(X)$, i.e. $xEy \Leftrightarrow s(x) = s(y)$. A set $Y \subseteq X$ is called a transversal for E if it meets every E-class at exactly one point, i.e. for each $x \in X$, $[x]_E \cap Y$ is a singleton.

Proposition 11.3. *An equivalence relation E on a Polish space X admits a Borel selector if and only if it admits an analytic transversal.*[13]

Proof. If $s : X \to X$ is a Borel selector for E, then it is clear that $s(X)$ is an analytic transversal. For the converse, let $Y \subseteq X$ be an analytic transversal and define $s : X \to X$ by $x \mapsto$ the unique $y \in Y$ with xEy. To prove that s is Borel, we fix a Borel set $B \subseteq X$ and show that $s^{-1}(B)$ is Borel. Note that $s^{-1}(B) = [B \cap Y]_E$ and hence is analytic. But also $(s^{-1}(B))^c = s^{-1}(B^c) = [B^c \cap Y]_E$, so $(s^{-1}(B))^c$ is also analytic, and thus $s^{-1}(B)$ is Borel. □

Thus, the chain of implications for general equivalence relations is as follows:

$$\text{Borel transversal} \Rightarrow \text{analytic transversal} \Leftrightarrow \text{Borel selector} \Rightarrow \text{smooth}.$$

Concerning the reverse direction of the first implication, we have the following:

Proposition 11.4. *For orbit equivalence relations of Borel actions of Polish groups, any analytic transversal is actually Borel.*

Proof. Let $G \curvearrowright X$ be a Borel action of a Polish group G on a Polish space X, and let $Y \subseteq X$ be an analytic transversal for E_G. Then, for $x \in X$,

$$x \notin Y \iff \exists g \in G \, (gx \in Y \text{ and } gx \neq x),$$

so Y^c is analytic as well, and hence Y is Borel. □

As for the implication "Borel selector $\Rightarrow$ smooth", it is a theorem of Burgess that the reverse implication is also true for the orbit equivalence relations of continuous actions of Polish groups.

11.B. **Examples of concrete classification.** We start by listing some well known examples of equivalence relations from different areas of mathematics that admit concrete classification.

Examples 11.5.

(a) *Isomorphism of finitely generated abelian groups.* Let $\mathcal{L}_g = \{\cdot, 1\}$ be the language of groups. Then the set $Y \subseteq X_{\mathcal{L}_g}$ of all finitely generated abelian groups is $\mathbf{\Sigma}^0_3$ ($\exists$ finitely many elements such that $\forall$ group elements γ $\exists$ a combination equal to γ), and hence standard Borel. We know from algebra that every $\Gamma \in Y$ is isomorphic to a group of the form $\mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \ldots \oplus \mathbb{Z}_{q_k}$, where $q_1 \leq q_2 \leq \ldots \leq q_k$ are powers of primes. The map

[13] Thanks to Aristotelis Panagiotopoulos for pointing out that assuming the existence of a merely analytic transversal still implies the existence of a Borel selector.

$\Gamma \mapsto \mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \ldots \oplus \mathbb{Z}_{q_k}$ from Y to Y is a selector for $\mathrm{Iso}(Y)$ and it can be shown to be Borel, witnessing the smoothness of $\mathrm{Iso}(Y)$.

(b) *Similarity of matrices.* Let $M_n(\mathbb{C})$ denote the Polish space of complex $n \times n$ matrices and $\sim$ denote the similarity relation on $M_n(\mathbb{C})$, which is $\boldsymbol{\Sigma}_1^1$ by definition. For each $A \in M_n(\mathbb{C})$, let $J(A)$ denote its Jordan canonical form. We know from linear algebra that $A \sim B \Leftrightarrow J(A) = J(B)$, in other words, J is a selector for $\sim$. Moreover, one can show that it is Borel, so $\sim$ is smooth. In particular, $\sim$ is a Borel equivalence relation, which wasn't apparent at all from its definition.

(c) *Isomorphism of Bernoulli shifts.* Let (X, μ) be a probability space (X can be finite) and let $\mu^{\mathbb{Z}}$ denote the product measure on $X^{\mathbb{Z}}$. Let $S : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ denote the shift automorphism, i.e. for $f \in X^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, $T(f)(n) = f(n-1)$. The dynamical system $(X^{\mathbb{Z}}, \mu^{\mathbb{Z}}, S)$ is called a *Bernoulli shift*. By the measure isomorphism theorem, every Bernoulli shift is isomorphic to $([0,1], \lambda, T)$, where λ is the Lebesgue measure and T some measure-preserving automorphism of $([0,1], \lambda)$. In this case, we would call T a Bernoulli shift as well, and let $B \subseteq \mathrm{Aut}([0,1], \lambda)$ be the set of all Bernoulli shifts. Ornstein showed that B is a Borel subset of $\mathrm{Aut}([0,1], \lambda)$, and hence is a standard Borel space. Furthermore, to each $T \in \mathrm{Aut}([0,1], \lambda)$, one can attach a real number $e(T) \in \mathbb{R} \cup \{\infty\}$ called the *entropy* of the dynamical system $([0,1], \lambda, T)$, which somehow measures the probabilistic unpredictability of the action of T. This notion of entropy is defined by Kolmogorov and it follows from the definition that it is an isomorphism invariant. For the Bernoulli shifts however (i.e. $T \in B$), it is a celebrated theorem of Ornstein that entropy is a complete invariant! In other words, for $T_1, T_2 \in B$, $([0,1], \lambda, T_1) \simeq ([0,1], \lambda, T_2) \Leftrightarrow e(T_1) = e(T_2)$. It can also be checked that the function $T \mapsto e(T)$ is Borel, hence a Borel reduction of the isomorphism relation of Bernoulli shifts to $\mathrm{Id}(\mathbb{R} \cup \{\infty\})$, witnessing the smoothness of the former.

11.C. **Nonsmooth equivalence relations.**

Definition 11.6. An equivalence relation E on a Polish space X (resp. measure space $(X, \mathcal{B}, \mu)$) is called *generically ergodic* (resp. *μ-ergodic*) if every invariant subset of X with the BP (resp. μ-measurable) is either meager (resp. μ-null) or comeager (resp. μ-conull).

We call a (continuous or measurable) group action $G \curvearrowright X$ generically ergodic (resp. μ-ergodic) if such is the induced orbit equivalence relation E_G.

Proposition 11.7. *Let E be an equivalence relation on a Polish space X and let $f : X \to 2^{\mathbb{N}}$ be a Baire measurable homomorphism of E to* $\mathrm{Id}(2^{\mathbb{N}})$. *If E is generically ergodic, then there is $y \in 2^{\mathbb{N}}$ such that $f^{-1}(y)$ is comeager. Letting μ be a Borel measure on X, the analogous statement holds for μ-ergodic E.*

Proof. We only prove the topological statement since the proof of the measure-theoretic statement is analogous. First note that for any $A \subseteq 2^{\mathbb{N}}$, $f^{-1}(A)$ is E-invariant by the virtue of f being a homomorphism. By recursion on n, we now define an increasing sequence $(s_n)_n \subseteq 2^{<\mathbb{N}}$ such that $|s_n| = n$ and $f^{-1}(N_{s_n})$ is comeager. Put $s_0 = \varnothing$, and suppose s_n is defined and satisfies the requirements. Since $f^{-1}(N_{s_n}) = f^{-1}(N_{s_n{}^\frown 0}) \cup f^{-1}(N_{s_n{}^\frown 1})$, for at least one $i \in \{0,1\}$, $f^{-1}(N_{s_n{}^\frown i})$ must be nonmeager, and hence comeager because $f^{-1}(N_{s_n{}^\frown i})$ is invariant and has the BP. Set $s_{n+1} = s_n{}^\frown i$. Having finished the construction of $(s_n)_n$, put $y = \bigcup_n s_n$. Then $f^{-1}(y) = f^{-1}(\bigcap_n N_{s_n}) = \bigcap_n f^{-1}(N_{s_n})$ is comeager. $\square$

Corollary 11.8. *Let E be an equivalence relation on a Polish space X. If E is generically ergodic (resp. ergodic) and every E-class is meager, then E is not smooth. Letting μ be a nontrivial Borel measure on X, the analogous statement holds for μ-ergodic E.*

Proof. If $f : X \to 2^{\mathbb{N}}$ is a Baire measurable reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$, then the preimage of every point $y \in f(X)$ is an E-class, and hence is meager, contradicting the previous proposition. □

This, together with (5) of Theorem 8.26, gives the following.

Corollary 11.9. *If a group Γ acts by homeomorphisms on a Polish space X such that every orbit is meager (e.g. when Γ is countable) and there is a dense orbit, then E_Γ is nonsmooth. In particular, if G is a Polish group and $\Gamma < G$ is a countable dense subgroup, then the coset equivalence relation E_Γ is nonsmooth.*

Examples 11.10.

(a) *The Vitali equivalence relation $\mathbb{E}_v$ is nonsmooth.* Indeed, $\mathbb{E}_v$ is the orbit equivalence relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$.

(b) *The irrational rotation $\mathbb{E}_\alpha$ of S^1 is nonsmooth.* Indeed, let Γ be the subgroup of S^1 generated by $e^{2\pi\alpha i}$. It is clear that $\mathbb{E}_\alpha$ is precisely the orbit equivalence relation induced by the translation action $\Gamma \curvearrowright S^1$, and it follows from irrationality of α that Γ is dense.

(c) *$\mathbb{E}_0$ is nonsmooth.* Indeed, each $\mathbb{E}_0$-class is countable dense and $\mathbb{E}_0$ is induced by a continuous action of a countable group as described in Example 9.5 (d). Moreover, like in the previous two examples, we can even view $\mathbb{E}_0$ as the orbit equivalence relation induced by the translation action of a countable dense subgroup $\Gamma < (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, namely, $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{\oplus\mathbb{N}} := \{q \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} : \forall^\infty n\ q(n) = 0\}$.

11.D. **The Harrington–Kechris–Louveau dichotomy.** For an equivalence relation E on a Polish space X, it is clear that if $\mathbb{E}_0 \leq_B E$ then E is nonsmooth. The following striking theorem shows that this is the only impediment to smoothness!

Dichotomy 11.11 (Harrington–Kechris–Louveau '90)**.** *For any Borel equivalence relation E on a Polish space X, either E is smooth, or else, $\mathbb{E}_0 \sqsubseteq_c E$.*

This theorem shows, in particular, that $\mathbb{E}_0$ is the $\leq_B$-minimum element (up to $\sim_B$) among all nonsmooth Borel equivalence relations; in other words, it is the minimum unsolvable classification problem among the Borel ones.

12. Countable Borel equivalence relations and actions of countable groups

Definition 12.1. An equivalence relation E on a Polish space X is called *countable* (resp. *finite*) if each E-class is countable (resp. finite).

12.A. **Arising from countable group actions.** Countable Borel equivalence relations naturally arise as orbit equivalence relations of Borel actions of countable groups; indeed, if a countable group Γ acts on a Polish space X in a Borel fashion, then the induced orbit equivalence relation E_Γ is countable and is Borel because for $x, y \in X$,

$$xE_\Gamma y \iff \exists \gamma \in \Gamma\,(\gamma \cdot x = y).$$

Surprisingly enough, this is the only way countable Borel equivalence relation arise, as the following important corollary of the Luzin–Novikov theorem shows:

Theorem 12.2 (Feldman–Moore)**.** *For any countable Borel equivalence relation E on a Polish space X, there is a Borel action $\Gamma \curvearrowright X$ of a countable group Γ with $E_\Gamma = E$. Moreover, Γ can be taken to be generated by involutions*[14] *so that for every $(x,y) \in E$, there is an involution $\gamma \in \Gamma$ with $\gamma \cdot x = y$.*

Proof. We only sketch the idea of the proof without going into details (see [Tse13, 22.2] for a complete proof). By Luzin–Novikov, $E = \bigsqcup_n \mathrm{graph}(f_n)$, where each f_n is a Borel partial function $X \rightharpoonup X$. Partition each $\mathrm{graph}(f_n)$ into countably many further Borel graphs of one-to-one Borel partial functions having their domain and range disjoint, so we can extend them to full Borel involutions of X and let Γ be the group (under composition) generated by these involutions. $\square$

12.B. **Smoothness for countable equivalence relations.** As mentioned above, it is a theorem of Burgess that smoothness is equivalent to admitting a Borel transversal/selector for the orbit equivalence relations of continuous actions of Polish groups. Here we will record a special case of this[15].

Proposition 12.3. *For a countable equivalence relation E on a Polish space X, the following are equivalent:*

(1) E is smooth.
(2) E admits a Borel selector.
(3) E admits a Borel transversal.
(4) E admits an analytic transversal.

Proof. The implication (1)⇒(2) follows immediately from Corollary 5.25. Similarly, (2)⇒(3) follows from Corollary 5.24. The rest was proven in Proposition 11.3. $\square$

Proposition 12.4. *Any finite Borel equivalence relation E on a Polish space X is smooth (equivalently, admits a Borel transversal).*

Proof. Let $<$ be a Borel linear ordering on X, which exists by Corollary 5.28. Put

$$A := \{x \in X : x = \min[x]_E\}.$$

It remains to show that A is Borel. But, for $x \in X$,

$$x \in A \iff \forall y \in X\,(yEx \Rightarrow x \leq y),$$

and the latter definition is Borel by Corollary 5.24. $\square$

[14] A group element $\gamma \in \Gamma$ is called an *involution* if $\gamma^2 = 1_\Gamma$, or equivalently, $\gamma^{-1} = \gamma$.

[15] The fact this is a special case is due to the Feldman–Moore theorem 12.2 and Corollary 5.17

12.C. **Complete sections.**

Definition 12.5. For an equivalence relation E on a set X, a set $A \subseteq X$ is called a *complete section for* E if it meets every E-class C, i.e. $A \cap C \neq \varnothing$.

In many proofs, it helps to have "small" Borel complete sections to mark points in each E-class that could guide us in our combinatorial constructions (like cairns on a hiking trail). Of course, the "smallest" Borel complete section is a transversal, but this only exists for smooth equivalence relations. However, even when E is nonsmooth, we can hope to take smaller and smaller complete sections, i.e. mark less and less points in each E-class. We make this precise here.

Definition 12.6. Let E be an equivalence relation on a Polish space X. A sequence $(B_n)_{n\in\mathbb{N}}$ of subsets of X is called a *marker sequence for* E if

(a) $(B_n)_n$ is decreasing;

(b) $(B_n)_n$ is vanishing, i.e. $\bigcap_n B_n = \varnothing$;

(c) each B_n is a complete section for E.

We call a marker sequence $(B_n)_n$ Borel if each B_n is Borel.

Note that for a marker sequence to exist, it is necessary for E to be *aperiodic*, i.e. each E-class must be infinite.

Marker (cairn?) Lemma 12.7. *Any aperiodic countable Borel equivalence relation E on a Polish space X admits a Borel marker sequence.*

Proof. If X is countable, every set is Borel, so the statement is trivial. Thus, by the Borel isomorphism, we may assume without loss of generality that $X = 2^{\mathbb{N}}$. This allows us to use the linear (lexicographic) ordering of $2^{\mathbb{N}}$, that's all.

For each $x \in X$, note that $[x]_E$ has a limit point because it is infinite and $2^{\mathbb{N}}$ is compact metrizable. Let $\mathrm{lp}(x)$ denote the leftmost (i.e. lexicographically least) limit point of $[x]_E$; note that $\mathrm{lp}(x)$ itself may or may not belong to $[x]_E$.[16]

Now for each n, put $x \in A_n$ if it is within 2^{-n} distance from $\mathrm{lp}(x)$, i.e.

$$x \in A_n \iff \mathrm{lp}(x)\restriction_n = x\restriction_n.$$

Claim. Each A_n is Borel.

Proof of Claim. Let $s_n : X \to 2^n$ be defined by $x \mapsto$ lexicographically least $s \in 2^n$ such that $|[x]_E \cap N_s| = \infty$ and note that $\mathrm{lp}(x)\restriction_n = s_n$, so it is enough to show that each s_n is Borel. To this end, observe that for $s \in 2^n$ and $x \in X$,

$$s_n(x) = s \iff |[x]_E \cap N_s| = \infty \text{ and } \forall t <_{\mathrm{lex}} s\ |[x]_E \cap N_t| < \infty,$$

and

$$|[x]_E \cap N_t| < \infty \iff \exists m \in \mathbb{N} \exists x_1, \dots, x_m \in E_x \cap N_t\, (x_1, \dots, x_m \in N_t \wedge \forall y \in E_x \cap N_t \bigvee_{i=1}^{m} y = x_i).$$

[16]It follows from the proof of the Marker Lemma that the function $\mathrm{lp} : X \to X$ is Borel, and note that it is a homomorphism of E to $\mathrm{Id}(X)$. However, it may not be a reduction since it is possible to have $[x]_E$ and $[y]_E$ disjoint yet $\mathrm{lp}(x) = \mathrm{lp}(y)$.

But checking existence in $E_x \cap N_t$ is Borel by Luzin–Novikov since $E_x \cap N_t = \big(E \cap (X \times N_t)\big)_x$ is countable for every $x \in X$. $\dashv$

The sequence $(A_n)_n$ is almost a marker sequence: it is decreasing and each A_n is clearly a complete section. However it may not be vanishing because some of the E-classes may actually contain their leftmost limit points, i.e. for some $x \in X$, it may be that $\mathrm{lp}(x) \in [x]_E$, in which case $\mathrm{lp}(x) \in A_n$ for all n. Note however, that these are the only possible points in $A := \bigcap_n A_n$, so A meets every E-class in at most one point. Thus, because for each $x \in X$, $[x]_E \cap A_n$ is infinite, $B_n := A_n \setminus A$ is still a complete section, but now we also have $\bigcap_n B_n = \varnothing$. $\square$

Corollary 12.8. *Any aperiodic countable Borel equivalence relation E on a standard probability space (X, μ) admits a Borel complete section of arbitrarily small measure.*

Proof. For any decreasing sequence $(B_n)_n$ the finiteness of μ and countable additivity guarantee downward continuity: $\lim_n \mu(B_n) = \mu(\bigcap_n B_n)$. $\square$

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