

Math 595 CET: CLASSIFICATION IN ERGODIC THEORY

Spring 2015, MWF 12:00–12:50pm in 241 Altgeld Hall

ANUSH TSERUNYAN

One of the main objects of study in ergodic theory is a probability-measure-preserving (p-m-p) dynamical system of the form (X, μ, T) , where (X, μ) is a standard probability space and T a p-m-p automorphism of (X, μ) , or more generally, (X, μ, Γ, α) , where α is a p-m-p action of a countable group Γ on (X, μ) . Naturally, we would like to classify these dynamical systems up to suitable notions of equivalence, such as isomorphism (conjugacy), unitary (spectral) equivalence, and orbit equivalence¹, by attaching invariants to these systems, such as entropy, spectral measures, cost. Here are some major positive results in this direction:

- (1) [Ornstein 1970] Two Bernoulli shifts (X, μ, T) and (Y, ν, S) are isomorphic if and only if they have equal entropy. [Halmos–von Neumann 1942] Two dynamical systems (X, μ, T) and (Y, ν, S) with discrete spectrum are isomorphic if and only if they are unitarily equivalent if and only if their sets of eigenvalues are equal.
- (2) [Dye 1963; Ornstein–Weiss 1980] Any two probability p-m-p actions of (maybe different) amenable groups are orbit equivalent.

Note that (1) classifies only special kinds of automorphisms up to isomorphism or unitary equivalence, leaving the general classification problem widely open. What if such classification was *impossible*? How would we *prove* this? This is where *descriptive set theory* enters the picture, providing a suitable framework and tools for proving non-classification results for equivalence relations.

The point of view taken here is *global*: we look at all p-m-p systems at once, i.e. we study the group $\text{Aut}(X, \mu)$ of all p-m-p automorphisms, as well as the space $\text{Act}(\Gamma, X, \mu)$ of all p-m-p actions of Γ on (X, μ) . Here are some striking victories of this new theory:

- (1') [Hjorth 2001; Foreman–Weiss 2004] Neither isomorphism, nor unitary equivalence, admits any “reasonable” classification even if we restrict to weakly mixing automorphisms.
- (2') [Epstein–Ioana–Kechris–Tsankov 2008] If Γ is a non-amenable countable group, then it admits continuum-many non-orbit-equivalent free p-m-p actions on (X, μ) . Moreover, orbit equivalence on $\text{Act}(\Gamma, X, \mu)$ does not admit any “reasonable” classification.

This course: Our goal is to cover or sketch most of what’s advertised above, as well as the basics of the theory of costs developed by Gaboriau and the rigidity phenomenon.

Prerequisites: Being comfortable with abstract measure theory, including L^2 -spaces, and, in general, Hilbert spaces. No knowledge of descriptive set theory is required as the necessary basics of it will be covered in the first week of the course.

Coursework: Each student will be asked to give one or two in-class presentations on relevant topics/papers that will be chosen based on the student’s interests.

References: Our main sources will be “Global Aspects of Ergodic Group Actions” by Kechris and “Topics in Orbit Equivalence” by Kechris–Miller.

Got questions? Email me at anush@illinois.edu.

¹For an action $\alpha : \Gamma \curvearrowright (X, \mu)$, let E_α be the *induced orbit equivalence relation* on X . Actions $\alpha : \Gamma \curvearrowright (X, \mu)$ and $\beta : \Delta \curvearrowright (Y, \nu)$ of countable groups Γ, Δ are called *orbit equivalent* if E_α and E_β are measure-isomorphic, i.e. there is a measure-isomorphism $T : (X, \mu) \xrightarrow{\sim} (Y, \nu)$ such that $x E_\alpha y \Leftrightarrow T(x) E_\beta T(y)$, for μ -a.e. $x, y \in X$.