MATH 570: MATHEMATICAL LOGIC

HOMEWORK 9

Due date: Nov 3 (Tue)

1. (a) Show that the Ackermann function grows faster than any primitive recursive function; more precisely, prove that for any primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists $n_f \in \mathbb{N}$ such that $f(\vec{x}) \leq A(n_f, \|\vec{x}\|_1)$ for all $\vec{x} \in \mathbb{N}^k$, where $\|\vec{x}\|_1 = x_1 + ... + x_n$.

Hint: Use the problem from the previous homework that had a list of properties of the Ackermann function.

(b) Conclude that the Ackermann function is not primitive recursive.

2. Prove the following proposition using the outline below.

Proposition. *There exists a recursive function* $\phi : \mathbb{N}^2 \to \mathbb{N}$ *such that for every* n, $\phi_n := \phi(n, \cdot)$ *is primitive recursive and for every* $k \in \mathbb{N}$ *and every primitive recursive function* $f : \mathbb{N}^k \to \mathbb{N}$, *there is* n *such that* $\forall \vec{a} \in \mathbb{N}^k$,

$$f(\vec{a}) = \phi_n(\langle \vec{a} \rangle).$$

In the latter case, we say that ϕ_n *corresponds to* f.

In this proposition and below, the tuple coding function $\langle \rangle$ is assumed to satisfy

$$\langle a_0, ..., a_{k-1} \rangle \geq \max \{k, a_0, ..., a_{k-1}\}.$$

Outline: Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be a function with the following property: for every $n \in \mathbb{N}$,

- if $n = \langle 0, a, m \rangle$ then ϕ_n corresponds to the successor function $S : \mathbb{N} \to \mathbb{N}$.
- if $n = \langle 1, a, m \rangle$ then ϕ_n corresponds to projection function $P_m^{(a)} : \mathbb{N}^a \to \mathbb{N}$.
- if $n = \langle 2, a, m \rangle$ then ϕ_n corresponds to constant function $C_m^{(a)} : \mathbb{N}^a \to \mathbb{N}$.
- if $n = \langle 3, a, m \rangle$, where
  - $m = \langle n_0, ..., n_k \rangle$ for some $k \geq 1$,
  - $(n_0)_1 = k$,
  - $(n_i)_1 = a$ for $i = 1, ..., k$,

  then, letting $g : \mathbb{N}^k \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h_i : \mathbb{N}^a \to \mathbb{N}$ the functions corresponding to ϕ_{n_i}, ϕ_n corresponds to the function obtained by *composition* from g and $h_0, ..., h_{k-1}$.
- if $n = \langle 4, a, m \rangle$, where
  - $a \geq 1$
  - $m = \langle n_0, n_1 \rangle$
  - $(n_0)_1 = a - 1$,
  - $(n_1)_1 = a + 1$,

  then letting $g : \mathbb{N}^{a-1} \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h : \mathbb{N}^{a+1} \to \mathbb{N}$ the function corresponding to ϕ_{n_1}, ϕ_n corresponds to the function obtained by *primitive recursion* from g and h.

Note that to calculate the value $\phi(n, l)$, one needs to know $\phi(n', l')$ for only finitely many (n', l') with either $n' < n$ or $l' < l$. Use this and an idea similar to the Dedekind analysis of recursion to show that ϕ is recursive (i.e. one can define a recursive ϕ satisfying the property above).

3. Prove the following:

(a) $\mathrm{Q} \nvDash (x + y) + z = x + (y + z)$;

(b) $\mathrm{Q} \nvdash x + y = y + x$;

(c) $\mathrm{Q} \nvdash \forall x(0 + x = x)$.

4. (a) Show that for a theory $T \subseteq \mathrm{Th}(\mathbf{N})$, the functions and relations representable in T are arithmetic. Conclude that the recursive functions and relations are arithmetical.

(b) Give a direct proof that the recursive functions/relations are arithmetical.

5. Show that Gödel's Incompleteness theorem (the original form, Theorem 6.2 in the notes) is equivalent to the statement that $\mathrm{Th}(\mathbf{N})$ is not recursive.