MATH 570: MATHEMATICAL LOGIC

HOMEWORK 6

Due date: Oct 6 (Tue)

1. Recall Problem 6* from HW2:

Show that the relation

$$P(x,y) \iff x \text{ and } y \text{ are connected}$$

is not definable in the undirected graph $\mathbf{\Gamma} := (\Gamma, E)$ that consists of two bi- infinite paths $\mathbb{Z}$.

Now that we have built some nontrivial machinery, try proving this again.

Hint: Let $A, B \subseteq \Gamma$ denote the two bi-infinite paths. Supposing that there is a τ_{graph}-formula $\varphi(x,y)$ defining the connectedness relation in $\mathbf{\Gamma}$, get an elementary extension of $\mathbf{\Gamma}$ containing two other bi-infinite paths C, D such that φ holds between elements of C and A, but φ doesn't hold between elements of D and A. Get a contradiction by swapping C and D.

2. The following is a well known theorem of additive combinatorics:

Theorem (van der Waerden)**.** *Suppose $\mathbb{N}$ is finitely colored. Then one of the color classes contains arbitrarily long arithmetic progressions.*

Use this theorem and the Compactness theorem to derive the following finitary version:

Theorem. *Given any positive integers m and k, there exists $N \in \mathbb{N}$ such that whenever $\{0, 1, ..., N-1\}$ is colored with m colors, one of the color classes contains an arithmetic progression of length k.*

3. (a) Show that the theory of 2-regular acyclic undirected graphs (i.e. every vertex has exactly two neighbors) is uncountably categorical and hence complete.

(b) (Anton Bernshteyn) Conclude, once again, that the class $\mathcal{D}$ of disconnected graphs is not axiomatizable.

4. Let DLO denote the theory of dense linear orderings without endpoints as defined in Example 5.2(b) of the lecture notes.

(a) Show that DLO is $\aleph_0$-categorical, and hence $(\mathbb{Q}, <)$ is the only (up to isomorphism) countable dense linear ordering without endpoints.

Hint: Enumerate the two models and recursively construct a sequence of finite partial isomorphisms by going back and forth between the models.

(b) Let $\tau_n := (<, c_1, ..., c_n)$, where c_i are constant symbols. Show that the theory

$$\text{DLO}_n = \text{DLO} \cup \{c_i < c_{i+1} : i < n\}$$

is $\aleph_0$-categorical. Conclude that DLO_n is complete.

(c) Let $\tau_\infty = (<, \{c_i : i \in \mathbb{N}\})$, where c_i are constant symbols. Show that the theory

$$\text{DLO}_\infty = \text{DLO} \cup \{c_i < c_{i+1} : i \in \mathbb{N}\}$$

is complete.

(d) Yet, show that DLO_∞ has exactly three countable nonisomorphic models and hence is not $\aleph_0$-categorical.