MATH 570: MATHEMATICAL LOGIC

HOMEWORK 5

Due date: Sep 29 (Tue)

Below, problems with $*$ are not mandatory.

1. Prove that a graph is 3-colorable[1] if and only if so is every finite subgraph of it.

Hint: To be able to express the 3-colorability property, add to the signature of graphs three unary relation symbols (one for each color). Also, add a name (constant symbol) for every vertex in the graph.

2. Show that the following classes of structures are not axiomatizable:

(a) cyclic groups,

Hint: Think about cardinalities of models of a hypothetical axiomatization.

(b) non-bipartite graphs,

Hint: Given a hypothetical axiomatization, add new sentences to it regarding nonexistence of certain cycles, so that the resulting theory is still finitely satisfiable, but its models are necessarily bipartite (actually acyclic) graphs.

(c) connected graphs.

Hint: Add two new constant symbols c_1, c_2 to the signature, and, given a hypothetical axiomatization, slowly add new sentences to it regarding the graph-distance between c_1 and c_2, so that the resulting theory is still finitely satisfiable, but its models are necessarily disconnected graphs.

3. Let $\mathbf{M} \vDash \mathsf{PA}$.

(a) Let $\mathbf{N} := (\mathbb{N}, 0^{\mathbf{N}}, S^{\mathbf{N}}, +^{\mathbf{N}}, \cdot^{\mathbf{N}})$ denote the standard model of PA, i.e. the usual natural numbers. Show that there is a unique homomorphism $f : \mathbf{N} \to \mathbf{M}$ and this f is one-to-one, and hence (why?) is a τ_{arthm}-embedding.

(b) In the notation of (a), define the standard part of $\mathbf{M}$ by

$$\overline{\mathbb{N}} = f[\mathbb{N}].$$

Show that if $\mathbf{M}$ is nonstandard then $\overline{\mathbb{N}}$ is not definable in $\mathbf{M}$.

(c) (Overspill) Assume $\mathbf{M}$ is nonstandard, and let $\phi(x, \vec{y})$ be a τ_{arthm}-formula, where $\vec{y}$ is a k-vector and $\vec{a} \in M^k$. Show that if $\mathbf{M} \vDash \phi(n, \vec{a})$ for infinitely many $n \in \overline{\mathbb{N}}$, then there is $w \in M \setminus \overline{\mathbb{N}}$ such that $\mathbf{M} \vDash \phi(w, \vec{a})$. In other words, if a statement is true about infinitely many natural numbers, then it is true about an infinite number.

4. Give an example of structures $\mathbf{A} \prec \mathbf{B}$ that demonstrate the failure of the converse of Problem 4 of HW3, which we quote here:

Problem 4 of HW3: Let $\mathbf{A} \subseteq \mathbf{B}$ and assume that for any finite $S \subseteq A$ and $b \in B$, there exists an automorphism f of $\mathbf{B}$ that fixes S pointwise (i.e. $f(a) = a$ for all $a \in S$) and $f(b) \in A$. Show that $\mathbf{A} \prec \mathbf{B}$.

[1]We say that a graph $\mathbf{\Gamma} = (V, E)$ is *k-colorable*, $k \in \mathbb{N}$, if it admits a *proper k-coloring*, i.e. a function $c : V \to \{0, 1, ..., k-1\}$, which assigns different values to adjacent vertices.

5. This problem illustrates the (crazy) structure of the nonstandard models of PA. Let $\mathbf{M}$ be a nonstandard model of PA and let $\overline{\mathbb{N}}$ be its standard part. By replacing it with $\mathbb{N}$, we can assume without loss of generality that $\overline{\mathbb{N}} = \mathbb{N}$.

(a) For $a, b \in M$, let $|a - b|$ denote the unique $d \in M$ such that $a + d = b$ or $b + d = a$ (why does such exist?). For all $a, b \in M$, define

$$a \sim b :\Leftrightarrow |a - b| \in \mathbb{N}.$$

Show that $\sim$ is an equivalence relation on M and that it is NOT definable in $\mathbf{M}$.

(b) Put $Q = M/\sim$, so $Q = \{[a] : a \in M\}$, where $[a]$ denotes the equivalence class of a. Define the relation $\leq_Q$ on Q as follows: for all $[a], [b] \in Q$,

$$[a] <_Q [b] :\Leftrightarrow \exists c \in M \smallsetminus \mathbb{N} \text{ such that } a + c = b$$

Show that $<_Q$ is well-defined (does not depend on the representatives a, b) and is a linear ordering on Q.

(c) Show that the ordering $(Q, \leq_Q)$ has a least element but no greatest element; moreover, show that it is a dense (in itself), i.e. for all $u, v \in Q$,

$$u <_Q v \Rightarrow \exists w (u <_Q w <_Q v),$$

where $u <_Q v$ stands for $u \leq_Q v \wedge u \neq v$.