MATH 570: MATHEMATICAL LOGIC

HOMEWORK 4

Due date: Sep 22 (Tue)

Below, problems with $*$ are not mandatory. You may use the Completeness and Compactness theorems starting from Problem 4.

1. Carefully prove the Constant Substitution lemma.

2. Pick two of the following statements and prove them. You may assume the preceding statements in your proofs.

(a) (Associativity of +) $\mathsf{PA} \vdash \forall x \forall y \forall z \big[(x+y)+z = x+(y+z)\big]$

(b) (Difference) $\mathsf{PA} \vdash \forall x \forall y \exists z \big[\phi(x,y,z) \wedge \forall u (\phi(x,y,u) \to u = z)\big]$, where $\phi \equiv x+z = y \vee y+z = x$.

(c) (0 is also a left-identity) $\mathsf{PA} \vdash \forall x (0 + x = x)$

(d) (Commutativity of +) $\mathsf{PA} \vdash \forall x \forall y (x+y = y+x)$

3. For a fixed signature τ, let $\mathcal{T}'$ denote the set of all (syntactically) consistent fully complete theories and we equip this set with the topology generated by the sets $\langle \phi \rangle := \{T \in \mathcal{T}' : T \vdash \phi\}$.

(a) Prove that this space is compact Hausdorff. (You shouldn't be using the Compactness theorem or any other big theorem.)

(b) Conclude that for $A \subseteq \mathcal{T}'$, if

$$A = \bigcap_{i \in I} \langle \phi_i \rangle = \bigcup_{j \in J} \langle \psi_j \rangle,$$

then there are finite $I_0 \subseteq I$, $J_0 \subseteq J$ such that

$$A = \bigcap_{i \in I_0} \langle \phi_i \rangle = \bigcup_{j \in J_0} \langle \psi_j \rangle.$$

In particular, the only clopen subsets of $\mathcal{T}'$ are of the form $\langle \phi \rangle$, where ϕ is a τ-sentence.

4. Show that if a theory has arbitrarily large finite models, then it has an infinite model.

5. (Weak Lefschetz Principle) Let ϕ be a τ_{ring}-sentence. Show that if $\mathsf{ACF}_0 \vDash \phi$, then for large enough primes p, $\mathsf{ACF}_p \vDash \phi$.

6. For a τ-theory T, let $\mathcal{M}_\tau(T)$ denote the class of its τ-models (i.e. nonempty τ-structures that satisfy it). Call T *finitely axiomatizable* if $\mathcal{M}_\tau(T)$ is finitely axiomatizable, i.e. there is a finite τ-theory S with $\mathcal{M}_\tau(S) = \mathcal{M}_\tau(T)$.

(a) For a τ-theory T, what does finite axiomatizability mean in terms of the topology (as in Problem 3)?

(b) Show that for every finitely axiomatizable theory T, there is a finite subset $T_0 \subseteq T$ with $\mathcal{M}_\tau(T_0) = \mathcal{M}_\tau(T)$.

7. A graph $\mathbf{\Gamma} = (V, E)$ is called *bipartite* if there is a partition $V = V_1 \uplus V_2$ into disjoint nonempty parts V_1, V_2, such that there is no edge between the vertices in the same part. Note that this definition is not first-order as it involves quantification over subsets of the underlying set.

(a) Nevertheless, show that the class of bipartite graphs is axiomatizable.

HINT: Find (google) an equivalent condition that is first-order.

(b) However, show that the class of bipartite graphs is not finitely axiomatizable.