MATH 570: MATHEMATICAL LOGIC

HOMEWORK 3

Due date: Sep 15 (Tue)

Below, problems with $*$ are not mandatory.

1. (Chasing definitions) A τ-theory T is said to be *semantically complete* if for any τ-sentence ϕ, $T \models \phi$ or $T \models \neg\phi$.

(a) Show that any satisfiable τ-theory T has a satisfiable full completion (i.e. a semantically complete τ-theory $\overline{T} \supseteq T$).

(b) Show that a τ-theory T is semantically complete if and only if for any $\mathbf{A}, \mathbf{B} \models T$, $\mathbf{A} \equiv \mathbf{B}$.

2. Let τ be a signature and show that the following are equivalent:

(a) For any τ-theory T, $T \models \phi$ implies that there is finite $T_0 \subseteq T$ with $T_0 \models \phi$.

(b) For any τ-theory T, T not satisfiable (semantically inconsistent) implies that some finite $T_0 \subseteq T$ is not satisfiable.

3. For a fixed signature τ, let $\mathcal{T}$ denote the set of all satisfiable fully complete theories and we equip this set with the topology generated by the sets $\langle\phi\rangle := \{T \in \mathcal{T} : T \models \phi\}$.

(a) Show that the sets $\langle\phi\rangle$ form an algebra. In particular, they form a basis for this topology, making it zero-dimensional[1] Hausdorff.

(b) Prove that the compactness of this space is equivalent to the statements in Problem 2.

Hint: Use the equivalent statement to compactness that involves closed sets, namely: A topological space is *compact* if and only if every family of closed sets with the finite intersection property[2] has a nonempty intersection.

4. Let $\mathbf{A} \subseteq \mathbf{B}$ and assume that for any finite $S \subseteq A$ and $b \in B$, there exists an automorphism f of $\mathbf{B}$ that fixes S pointwise (i.e. $f(a) = a$ for all $a \in S$) and $f(b) \in A$. Show that $\mathbf{A} \prec \mathbf{B}$.

5. Show that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$, but $(\mathbb{Q}, <) \not\simeq (\mathbb{R}, <)$.

Hint: Use the previous problem.

6. (Anton Bernshteyn, 2014) Let $\mathbf{B} = (B, E)$ be a countable undirected graph, each of whose vertices have degree at most 1. Suppose further that $\mathbf{B}$ has infinitely many vertices of degree 0 and infinitely many of degree 1. Find a substructure $\mathbf{A} \subseteq \mathbf{B}$ that is isomorphic to $\mathbf{B}$ and yet is not an elementary substructure of $\mathbf{B}$.

Remark: Note that $\mathbf{A} \simeq \mathbf{B}$ implies $\mathbf{A} \equiv \mathbf{B}$. Thus, this is an example of a substructure that is elementarily equivalent to the larger structure and yet isn't an elementary substructure.

[1] A topology is called *zero-dimensional* if it has a basis consisting of clopen (i.e. both closed and open) sets.

[2] A family $\mathcal{F}$ of sets is said to have the *finite intersection property* if for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$, $\bigcap_{A \in \mathcal{F}_0} A \neq \varnothing$.

7. Conclude from the Löwenheim-Skolem theorem that any satisfiable theory T has a model of cardinality at most $\max\{|\tau|, \aleph_0\}$. In particular, if ZFC is satisfiable, then it has a countable model (although that model $\mathbf{M}$ would believe it contains sets of uncountable cardinality, e.g. $\mathbb{R}^{\mathbf{M}}$). Explain why this DOES NOT imply that ZFC is not satisfiable.

8.*

(a) Find a τ_{graph}-sentence φ, all of whose models are 2-colorable undirected graphs with the degree of every vertex being at least 4, and such that φ has infinite models.

Hint: Demand existence of two "special" vertices.

(b) Find a τ_{graph}-sentence φ, all of whose models are infinite undirected graphs.

Hint: Demand existence of three "special" vertices, which allow to define a function on the set of vertices. Demand that this function is injective but not surjective.

Remark: You don't need to know any graph theory to solve this problem.