MATH 570: MATHEMATICAL LOGIC

HOMEWORK 2

Due on Sep 8 (Tue)

Below, problems with $*$ are not mandatory (for fun).

1. (a) A formula is called *universal* (resp. *existential*) if it is of the form $\forall x_1 \forall x_2 ... \forall x_n \psi$ (resp. $\exists x_1 \exists x_2 ... \exists x_n \psi$), where ψ is quantifier free. Let $\mathbf{A}, \mathbf{B}$ be τ-structures with $\mathbf{A} \subseteq \mathbf{B}$ and let $\phi(\vec{v})$ be a τ-formula. Show that for any $\vec{a} \in A^n$,

- if ϕ is quantifier free, then $\mathbf{A} \vDash \phi(\vec{a}) \iff \mathbf{B} \vDash \phi(\vec{a})$;
- if ϕ is universal, then $\mathbf{B} \vDash \phi(\vec{a}) \implies \mathbf{A} \vDash \phi(\vec{a})$;
- if ϕ is existential, then $\mathbf{A} \vDash \phi(\vec{a}) \implies \mathbf{B} \vDash \phi(\vec{a})$.

(b) Find a sentence that is true in $(\mathbb{N}, <)$ but false in $(\mathbb{Z}, <)$.

2. Let $\vec{v_1}, ..., \vec{v_n} \in \mathbb{Q}^m$. Show that $\{\vec{v_1}, ..., \vec{v_n}\}$ is linearly independent over $\mathbb{Q}$ if and only if it is linearly independent over $\mathbb{R}$.

Hint: Show that linear independence can be expressed by both universal and existential formulas.

3. Let ϕ be a τ-sentence. The *finite spectrum* of ϕ is the set

$$\{n \in \mathbb{N}^+ : \text{there is } \mathbf{M} \vDash \phi \text{ with } |M| = n\},$$

where $\mathbb{N}^+$ is the set of positive integers.

(a) Let $\tau = (E)$, where E is a binary relation symbol, and let ϕ be the sentence that asserts that E is an equivalence relation each class of which has exactly 2 elements. Show that the finite spectrum of ϕ is all positive even numbers.

(b) For each of the following subsets of $\mathbb{N}^+$, show that it is the finite spectrum of some sentence ϕ in some signature τ:

(i) $\{2^n 3^m : n, m \in \mathbb{N}^+\}$,

Hint: Groups...

(ii) $\{n \in \mathbb{N}^+ : n \text{ is composite}\}$,

Hint: Subgroups and cosets... (Although, there are other ways to do this.)

(iii) $\{p^n : p \text{ is prime and } n \in \mathbb{N}^+\}$,

Hint: Fields...

(iv) $\{n^2 : n \in \mathbb{N}^+\}$,

Hint: One way to do this is using the signature (S, f, g), where S is a unary relation symbol (hence will be interpreted as a subset of the universe), and f, g are unary function symbols.

(v)* $\{p : p \text{ is prime}\}$.

Hint: Fields with an ordering... (Although, there are other ways to do this.)

4. Let $\mathbf{A}$ be a τ-structure and $P \subseteq A$. Show that any automorphism $h : \mathbf{A} \xrightarrow{\sim} \mathbf{A}$ that fixes P pointwise (i.e. for every $p \in P$, $h(p) = p$) must fix every P-definable set $D \subseteq A^n$ setwise (i.e. $h(D) = D$, where, as usual, $h(\vec{d}) = (h(d_1), ..., h(d_n))$ for $\vec{d} \in D$).

5. Determine whether the following are 0-definable:

(a) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.
    Hint: You need a nontrivial fact from elementary number theory.
(b) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.
(c) The set of non-negative numbers in $(\mathbb{Q}, +)$.
(d) The set of positive numbers in $(\mathbb{R}, <)$.
(e) The function $\max(x, y)$ in $(\mathbb{R}, <)$.
(f) The function $\text{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.
(g) 2 in $(\mathbb{R}, +, \cdot)$.
(h) The relation $d(x, y) = 2$ in an undirected graph (with no loops) (Γ, E), where $d(x, y)$ denotes the edge distance function.

Hint: Use the previous problem to prove the negative answers.

6*. Show that the relation

$$P(x, y) \iff x \text{ and } y \text{ are connected}$$

is not 0-definable in the undirected graph $\mathbf{\Gamma} := (\Gamma, E)$ that consists of two bi-infinite paths, i.e. two disjoint copies of $\mathbb{Z}$.

Hint: Here is a solution suggested by Anton Bernshteyn. Show that for any τ_{graph}-formula $\phi(\vec{x}, \vec{y})$ with $\vec{x} = (x_1, ..., x_n)$ and $\vec{y} = (y_1, ..., y_k)$, there is a τ_{graph}-formula $\psi(\vec{x}, \vec{y})$ that is a Boolean combination of formulas that depend either only on $\vec{x}$ or only on $\vec{y}$ (i.e. are either of the form $\theta(\vec{x})$ or $\theta(\vec{y})$) such that for some $D \geq 1$, for all $\vec{a} \in \Gamma^n, \vec{b} \in \Gamma^k$ with $d_e(\vec{a}, \vec{b}) \geq D$,

$$\mathbf{\Gamma} \models \phi(\vec{a}, \vec{b}) \Leftrightarrow \mathbf{\Gamma} \models \psi(\vec{a}, \vec{b}).$$

When handling the case of the quantifier $\exists$, use the disjunctive normal form (google it, if you haven't seen it) to distribute the $\exists$ through Boolean combinations.