MATH 570: MATHEMATICAL LOGIC

HOMEWORK 13

Due date: Dec 8 (Tue)

1. Call a set $D \subseteq \mathbb{Z}$ a *difference set*[1] or a Δ-*set* if there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ of pairwise distinct elements such that $D := \{z_n - z_m : n > m\}$. Show that Δ-sets enjoy the *Ramsey property*; namely, for any $D \in \Delta$, whenever D is partitioned into two sets, at least one of them contains a Δ-set. Conclude that Δ^* is a filter.

2. Prove the Compactness Theorem using ultraproducts as follows. Let T be a set of τ-sentences and suppose that it is finitely satisfiable, i.e. every finite subset has a model. To warm up, first assume T is countable and construct a model of T as an ultraproduct over any nonprincipal ultrafilter on T. For the general case, let $I = \mathscr{P}_{\text{fin}}(T)$ be the set of all finite subsets of T and take an ultrafilter on I that contains all of the *cones*, i.e. the sets of the form $C_\phi := \{F \in I : \phi \in F\}$, for $\phi \in T$.

3. Let X be a topological space and α an ultrafilter on X. Call $x \in X$ a *limit point* of α if every open neighborhood of x belongs to α (has measure 1). Prove the following characterizations of Hausdorffness and compactness.

(a) X is Hausdorff if and only if every ultrafilter on X has at most one limit point.

(b) X is compact if and only if every ultrafilter on X has at least one limit point.

Hint: For $\Rightarrow$, prove the contrapositive, and for $\Leftarrow$, show that any collection of closed sets with the finite intersection property has a nonempty intersection.

4. Let $\mathbf{M}$ be a λ-saturated τ-structure. Prove that if

$$B = \bigcap_{i\in I} C_i = \bigcup_{j\in J} D_j,$$

for some definable (with parameters) sets $C_i, D_j \subseteq M^n$ and $|I|, |J| < \lambda$, then there exists finite $I_0 \subseteq I$, $J_0 \subseteq J$ such that

$$B = \bigcap_{i\in I_0} C_i = \bigcup_{j\in J_0} D_j.$$

[1] Note that the definition here is slightly different than the one given in class; this one is the correct definition.