MATH 570: MATHEMATICAL LOGIC

HOMEWORK 12

Due date: Dec 1 (Tue)

1. For $A \subseteq \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, we say that f *enumerates* A if the image of f is A, i.e. $f(\mathbb{N}) = A$. A set $A \subseteq \mathbb{N}$ is called *recursively enumerable* (*r.e.* for short) if there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ enumerating A.

Prove the following characterizations of recursive and Σ_1^0 sets:

(a) A set $A \subseteq \mathbb{N}$ is recursive if and only if it is either finite or enumerable by a *strictly increasing* recursive function.

(b) For a nonempty set $A \subseteq \mathbb{N}$, the following are equivalent:
(1) A is Σ_1^0
(2) A is either finite or enumerable by an *injective* recursive function
(3) A is r.e.

2. (William Balderrama) Let τ be a finite signature and T a τ-theory. If T admits q.e. and is decidable (e.g. when complete), then it actually admits effective q.e.

3. Show that $(\mathbb{N}, 0, S)$ admits effective quantifier elimination. Conclude that the only definable sets are the finite and cofinite sets.

4. Let $\tau = (0, 1, +, -, \cdot, <)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, -, \cdot, <)$.

(a) Show that for every subset $S \subseteq \mathbb{Q}$ that is definable in $\mathbf{Q}$ by a quantifier free formula, there is $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \varnothing$.

HINT: Prove this by induction on the construction (length) of the quantifier-free formula defining S.

(b) Use (a) to show that $\mathrm{Th}(\mathbf{Q})$ does NOT admit quantifier elimination.

5. Show that the theory of vector spaces over $\mathbb{Q}$ is diagram-complete and hence admits q.e. Conclude that the only definable sets in any $\mathbb{Q}$-vector space are the finite and cofinite sets.

6. Let $\tau = (0, 1, +, -, \cdot)$ and $\mathbf{R} = (\mathbb{R}, 0, 1, +, -, \cdot)$.

(a) Using Tarski's theorem about $(\mathbb{R}, 0, 1, +, -, \cdot, <)$, prove that $\mathbf{R}$ is model-complete.

(b) Show that every subset $S \subseteq \mathbb{R}$ that is definable in $\mathbf{R}$ by a q.f. formula is either finite or cofinite.

(c) Deduce that $\mathbf{R}$ does not admit q.e.