MATH 570: MATHEMATICAL LOGIC

HOMEWORK 11

Due date: Nov 17 (Tue)

1. (a) Show that the set of Σ_1^0 relations is closed under finite unions/intersections and projections, i.e. under the operations $\vee, \wedge, \exists$.

(b) Conclude that the set of Π_1^0 relations is closed under finite unions/intersections and co-projections, i.e. under the operations $\vee, \wedge, \forall$.

2. Let T be a τ-theory, where τ is a finite signature.

(a) Prove that if T is decidable, then it has a recursive completion.[1]

(b) Deduce **Church's theorem**: Any τ-theory that interprets Q is undecidable.

3. (a) Show that $U_Q^{(k)}$ (or $U_{\mathrm{PA}}^{(k)}$, if you prefer) is a $\Sigma_1^0(\mathbb{N}^k)$-universal relation.

(b) For which more general class of theories does your proof work? E.g. would it work for $T := \mathrm{PA} \cup \{\neg\gamma\}$, where γ is the Gödel sentence?

4. (a) Observe that the universal Σ_1^0 relation $U_Q(\cdot, \cdot_k) \subseteq \mathbb{N}^{1+k}$ from Proposition 7.12 is of the form $\exists y P(\cdot, \cdot_k, y)$, where $P \subseteq \mathbb{N}^{1+k+1}$ is primitive recursive; more precisely, for all $a \in \mathbb{N}, \vec{b} \in \mathbb{N}^k$,

$$U_Q(a, \vec{b}) \iff \exists y P(a, \vec{b}, y).$$

(b) Conclude **Kleene's Normal Form theorem**, namely: Every Σ_1^0 relation $R(\cdot_k) \subseteq \mathbb{N}^k$ is of the form $\exists y P(\cdot_k, y)$ for some primitive recursive relation P.

5. (a) Prove **Craig's lemma**, namely:, Any Σ_1^0 theory T (in a finite signature τ) has a recursive axiomatization.

Hint: For a sentence ϕ, ϕ being in T is "recursively witnessed" by a number $n_\phi \in \mathbb{N}$. Modify ϕ into a logically equivalent sentence that encodes the witness n_ϕ.

(b) Conclude that we can replace "recursive" by "Σ_1^0" in Rosser's form of the First Incompleteness theorem.

Remark: Recall that we couldn't replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.

(c) Conclude further that, in fact, any Σ_1^0 theory has a primitive recursive axiomatization.

Hint: Use Kleene's Normal Form.

[1] Many thanks to Anton Bernshteyn for suggesting this question.