MATH 570: MATHEMATICAL LOGIC

HOMEWORK 10

Due date: Nov 10 (Tue)

1. (a) Show that we can replace "recursive" by "arithmetical" in the statement of Gödel's Incompleteness theorem (the original form), i.e. prove that if $T \subseteq \mathrm{Th}(\mathbf{N})$ is arithmetical, then it is incomplete.

(b) Show that there exists an arithmetical completion of PA, i.e. there is a complete τ_{arthm}-theory $T \supseteq \mathrm{PA}$ such that $\ulcorner T \urcorner = \{\ulcorner \phi \urcorner : \phi \in T\}$ is an arithmetical subset of $\mathbb{N}$. Conclude that we CANNOT replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.

HINT: Mimic the inductive version of the proof that any theory has a (syntactically) consistent completion.

2. For each of the following statements, prove or give a counter-example to the assertion that it is true for *every* τ_{arthm}-sentence θ:

(a) $\mathrm{PA} \vdash \theta \iff \mathbf{N} \models \mathbf{Provable}_{\mathrm{PA}}([\theta])$,

(b) $\mathrm{PA} \vdash \theta \to \mathbf{Provable}_{\mathrm{PA}}([\theta])$,

(c) $\mathrm{PA} \vdash \theta \implies \mathrm{PA} \vdash \mathbf{Provable}_{\mathrm{PA}}([\theta])$.

3. Let ϕ and θ be τ_{arthm}-sentences. Consider the following statements:

1 $\mathrm{PA} \vdash \phi \implies \mathrm{PA} \vdash \theta$;

2 $\mathrm{PA} \vdash \phi \to \theta$.

Are they equivalent for all ϕ, θ? If not, which implication holds and which may fail? Prove your answers.

4. For each of the following τ_{arthm}-sentences, prove or give a counter-example to the assertion that it is provable in PA for *every* τ_{arthm}-sentence θ:

(a) $\mathbf{Provable}_{\mathrm{PA}}([\theta]) \to \theta$

(b) $\mathbf{Provable}_{\mathrm{PA}\cup\{\neg\theta\}}([\theta]) \to \mathbf{Provable}_{\mathrm{PA}}([\theta])$

(c) $\mathbf{Provable}_{\mathrm{PA}}([\theta]) \to \neg\mathbf{Provable}_{\mathrm{PA}}([\neg\theta])$

(d) $\mathbf{Provable}_{\mathrm{PA}}(\mathbf{Provable}_{\mathrm{PA}}([\theta])) \to \mathbf{Provable}_{\mathrm{PA}}([\theta])$

5. Let τ be a finite signature. State and prove Loeb's theorem for any recursive τ-theory T that interprets PA.