MATH 570: MATHEMATICAL LOGIC, FALL 2015

HOMEWORK 1

Due date: not sure yet, no earlier than next Tuesday (Sep 1)

1. Define appropriate signatures for vector spaces over $\mathbb{Q}$.

2. If $h : \mathbf{A} \to \mathbf{B}$ is a τ-homomorphism then the image $h(A)$ is a universe of a substructure of $\mathbf{B}$.

3. Prove that if τ does not contain any relation symbols, then any bijective τ-homomorphism is a τ-isomorphism. (That's why this happens with groups and rings, but not with graphs or orderings.)

4. A structure is called *rigid* if it has no automorphisms[1] other than the identity. Show that the structures $\mathbf{N} = (\mathbb{N}, 0, S, +, \cdot)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, \cdot)$ are rigid.

5. A structure $\mathbf{M}$ is called *ultrahomogeneous* if given any isomorphism between two finitely generated substructures, it extends to an automorphism of the whole structure, i.e. if $\mathbf{A}, \mathbf{B}$ are finitely generated substructures of $\mathbf{M}$ and $h : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then there is an automorphism $\overline{h}$ of $\mathbf{M}$ with $\overline{h} \sqsupseteq h$. Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. The same proof should also work to show that $(\mathbb{R}, <)$ is ultrahomogeneous.

6. For τ-structures $\mathbf{A}, \mathbf{B}$, we write $\mathbf{A} \equiv \mathbf{B}$ if for every τ-sentence ϕ, $\mathbf{A} \models \phi \iff \mathbf{B} \models \phi$.

(a) For a fixed *finite* group $\mathbf{G}$ (as a τ_{group}-structure), show that there is a τ_{group}-sentence ϕ such that for any τ_{group}-structure $\mathbf{M}$,

$$\mathbf{M} \models \phi \iff \mathbf{M} \simeq \mathbf{G}.$$

(b) More generally, let τ be a *finite* signature and $\mathbf{A}$ be a *finite* τ-structure. Show that there is a τ-sentence ϕ such that for any τ-structure $\mathbf{B}$,

$$\mathbf{B} \models \phi \iff \mathbf{B} \simeq \mathbf{A}.$$

In particular,

$$\mathbf{B} \equiv \mathbf{A} \iff \mathbf{B} \simeq \mathbf{A}.$$

[1] isomorphism with itself