MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

HOMEWORK 5

Due on Wednesday, Oct 7

Exercises from the textbook. 3.41, 3.44[1], 4.11, 4.12, 4.20(a)(b), 4.29, 4.31

Out-of-the-textbook exercises (these are as mandatory as the ones from the textbook)**.**

1. Prove that for any $a, b \in \mathbb{R}$,

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^{n-1-i}b^i,$$

with the convention that any number (including 0) to the power 0 is 1.

HINT: Prove this by induction on n. To make sure you don't get confused with indices of the summations, replace the $\sum$ notation with the usual (more informal) notation:

$$\sum_{i=0}^{n-1} a^{n-1-i}b^i = a^{n-1} + a^{n-2}b^1 + a^{n-3}b^2 + \ldots + a^2b^{n-3} + a^1b^{n-2} + b^{n-1}.$$

2. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers satisfying

$$a_0 = 0, a_1 = 1$$
$$a_{n+2} = a_{n+1} + a_n.$$

This is the well-known Fibonacci sequence. Prove that for every $n \geq 0$,

$$a_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},$$

where φ and ψ are the two distinct solutions to $x - 1 = \frac{1}{x}$; in other words, φ is the *golden ratio* $\frac{1+\sqrt{5}}{2}$ and ψ is its conjugate $\frac{1-\sqrt{5}}{2}$.

HINT: First, write the proposed expression for a_n using only numbers, then prove it by strong induction.

3. For each of the following functions, determine whether it is injective/surjective/bijective.

(a) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 3n - 2$.

(b) $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) := \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}.$$

(c) $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(m) := m^2 + 3$

4. Let O and E denote the sets of odd and even natural numbers, respectively. Find bijections between the following sets:

[1] HINT FOR 3.44: The main point is that starting from 18, all numbers are of this form. Prove this by strong induction.

(a) O and $\mathbb{N}$;

(b) E and $\mathbb{N}$;

(c) O and E.

Make sure to prove that the functions you define are indeed bijections by either showing that they are injective and surjective, or that they admit inverse functions.