Math 574: SET THEORY
MWF 3:00pm-3:50pm

ANUSH TSERUNYAN

Do you want to

- properly learn the axioms of ZFC, ordinals, cardinals and transfinite induction?
- not roll your eyes when someone says "it's not a set, it's a proper class"?
- understand the hierarchy of Borel sets in and out?
- embed the Cantor set into any uncountable Borel set?
- play infinite games on sets of reals to show that they are nice (e.g. measurable)?
- master the technique of Baire category and understand statements like "the generic compact set is perfect" or "the generic measure-preserving automorphism is ergodic"?
- color Borel graphs or show that they cannot be colored with small number of colors?
- define what it means for a classification problem in mathematics to be "hopeless" and prove that there is a *minimum* "hopeless" problem that sits inside every other "hopeless" problem?

Then you need to learn **Descriptive Set Theory** and this course may be for you.

What is Descriptive Set Theory? Descriptive set theory (DST) combines techniques from set theory, topology, analysis, recursion theory and other areas of mathematics to study definable subsets of $\mathbb{R}$ or, more generally, of any Polish space. Examples of such sets include Borel, analytic (projections of Borel), co-analytic (complement of analytic), etc. The framework of Polish spaces being used is justified by its robustness since, by Kuratowski's theorem, Polish spaces of the same cardinality are Borel isomorphic. A typical example (one of the first) of a theorem in DST is Souslin's theorem that states that if a set is both analytic and co-analytic, then it is Borel. At its earlier stage, a central interest in DST was investigating the regularity properties of definable sets such as the perfect set property (being countable or containing a perfect set, a version of Continuum Hypothesis that Cantor proved for closed sets), measurability and the Baire property. As it turned out, all these properties are satisfied by analytic sets, but curiously enough, whether they hold for the projections of co-analytic sets is already independent from ZFC.

For the past twenty years, a major focus of descriptive set theory has been the study of equivalence relations on Polish spaces that are definable when viewed as sets of pairs (e.g. orbit equivalence relations of continuous actions of Polish groups are analytic). This study is motivated by foundational questions such as understanding the nature of complete classification of mathematical objects (measure preserving transformations, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, etc.) and creating a mathematical framework for measuring the complexity of such classification problems. Due to its broad scope, it has natural interactions with other areas of mathematics, such as ergodic theory and topological dynamics, functional analysis and operator algebras, representation theory, topology, model theory and recursion theory.

About the course. We will cover the basics of ZFC in the first couple of weeks, and the rest of the course will be devoted to *descriptive set theory*, addressing everything advertised above and culminating in a proof of the powerful Kechris-Solecki-Todorcevic dichotomy about colorings of Borel graphs, as well as a proof (in a special case) of the striking Harrington-Kechris-Louveau dichotomy, which exhibits a minimum element among all classification problems that are "hopeless".

No textbook is needed, we will use my lecture notes that will be posted on my webpage. However, having Kechris's *Classical Descriptive Set Theory* at hand will be useful.