# DESCRIPTIVE SET THEORY

HOMEWORK 9

Due on Tuesday, Apr 8

**All problems below are to be presented.**

**1.** (a) Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets, i.e. for a given Polish space $(X, \mathcal{T})$, there exists a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.

(b) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be Polish and $f : X \to Y$ a Borel isomorphism. Show that there are Polish topologies $\mathcal{T}'_X \supseteq \mathcal{T}_X, \mathcal{T}'_Y \supseteq \mathcal{T}_Y$ with $\mathcal{B}(\mathcal{T}'_X) = \mathcal{B}(\mathcal{T}_X)$, $\mathcal{B}(\mathcal{T}'_Y) = \mathcal{B}(\mathcal{T}_Y)$ such that $f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}'_Y)$ is a homeomorphism. Moreover, $\mathcal{T}'_X, \mathcal{T}'_Y$ can be taken to be zero-dimensional.

(c) Let $G$ be a countable group and consider a Borel action of $G$ on a Polish space $(X, \mathcal{T})$, i.e. each $g \in G$ acts as a Borel automorphism of $X$. Prove that there exists a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$ that makes the action of $G$ continuous. Moreover, $\mathcal{T}_0$ can be taken to be zero-dimensional.

**2.** Show that the class of analytic sets is closed under

(a) continuous preimages,

(b) continuous images,

(c) countable unions,

(d) countable intersections.

Can we replace "continuous" by "Borel" above?

**3.** Let $X$ be Polish and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of disjoint analytic sets in $X$. Prove that there are disjoint Borel sets $\{B_n\}_{n \in \mathbb{N}}$ with $B_n \supseteq A_n$.

**4.** Let $X, Y$ be Polish and $f : X \to Y$ Borel. Show that for $A \subseteq f(X)$, if $f^{-1}(A)$ is Borel, then $A$ is Borel relative to $f(X)$, i.e. there is a Borel $A' \subseteq Y$ such that $A = A' \cap f(X)$.

**5.** Let $X$ be Polish and let $E$ be an analytic equivalence relation on $X$ (i.e. $E \subseteq X^2$ is analytic).

(a) Show that for an analytic set $A$, its saturation $[A]_E = \{x \in X : \exists y \in A (xEy)\}$ is also analytic.

(b) Let $A, B \subseteq X$ be disjoint invariant analytic sets (i.e. $[A]_E = A$, $[B]_E = B$). Prove that there is an invariant Borel set $D$ separating $A$ and $B$, i.e. $D \supseteq A$ and $D \cap B = \varnothing$.

**6.** Construct an example of a closed equivalence relation $E$ on a Polish space $X$ and a closed set $C \subseteq X$ such that the saturation $[C]_E$ is analytic but not Borel.

Remark: This shows that in part (a) of the previous problem, "analytic" is the best we can hope for.

Hint: Take analytic $A \subseteq \mathcal{N}$ that's not Borel and let $C \subseteq \mathcal{N}^2$ be a closed set projecting down onto $A$. Define an appropriate equivalence relation $E$ on $\mathcal{N}^2$ (i.e. $E \subseteq \mathcal{N}^2 \times \mathcal{N}^2$).