DESCRIPTIVE SET THEORY

HOMEWORK 8

Due on Tuesday, Apr 1 (no joke)

All problems below are to be presented.

1. Prove the following facts about the density topology on $\mathbb{R}$. (Below λ denotes the Lebesgue measure on $\mathbb{R}$ and all topological terms are with respect to the density topology.)

(a) Every nonempty open set has positive measure.

(b) For a Lebesgue measurable set $A \subseteq \mathbb{R}$, explicitly compute $\mathrm{Int}(A)$ and $\overline{A}$, and conclude that $\lambda(\mathrm{Int}(A)) = \lambda(A) = \lambda(\overline{A})$.

2. Consider $\mathbb{R}$ with the density topology and Lebesgue measure λ. For $A \subseteq \mathbb{R}$, prove that the following are equivalent:

(1) A is nowhere dense in the density topology;
(2) A is meager in the density topology;
(3) A is λ-null.

Conclude that A has the BP in the density topology if and only if it is Lebesgue measurable.

3. Let $X = \mathbb{I}^{\mathbb{N}}$ and put $C_0 = \{(x_n)_{n\in\mathbb{N}} : x_n \to 0\}$. Show that C_0 is in $\mathbf{\Pi}^0_3(X)$.

4. Let X be a topological space, $Y \subseteq X$, and let ξ be an ordinal with $1 \le \xi < \omega_1$. Prove the following:

(a) If $\mathbf{\Gamma}$ is one of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathcal{B}$, then $\mathbf{\Gamma}(Y) = \mathbf{\Gamma}(X)\downharpoonright_Y := \{A \cap Y : A \in \mathbf{\Gamma}(X)\}$.

(b) We also always have $\mathbf{\Delta}^0_\xi(Y) \supseteq \mathbf{\Delta}^0_\xi(X)\downharpoonright_Y$. If moreover, $Y \in \mathbf{\Delta}^0_\xi(X)$, then we also have $\mathbf{\Delta}^0_\xi(Y) \subseteq \mathbf{\Delta}^0_\xi(X)\downharpoonright_Y$. However, give an example of a Polish space X and $Y \subseteq X$ such that the last inclusion is false for $\xi = 1$.

5. A class $\mathbf{\Gamma}$ of sets is called self-dual if it is closed under complements, i.e. $\check{\mathbf{\Gamma}} = \mathbf{\Gamma}$. Show that if $\mathbf{\Gamma}$ is a self-dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set for $\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$, can have X-universal sets.

6. Letting X be a separable metrizable space and $\lambda < \omega_1$ be a limit ordinal, put

$$\mathbf{\Omega}^0_\lambda(X) := \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \ \left(= \bigcup_{\xi<\lambda} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(X)\right).$$

(a) Let Y be an uncountable Polish space and prove that there exists a set $P \subseteq \mathbf{\Delta}^0_\lambda(Y \times X)$ that parameterizes $\mathbf{\Omega}^0_\lambda(X)$.

Hint: First construct such a set for $Y = \mathbb{N} \times \mathcal{C}$. Then conclude it for $Y = \mathcal{C}$ using the fact that the following functions are continuous: $()_0 : \mathcal{C} \to \mathbb{N}$ and $()_1 : \mathcal{C} \to \mathcal{C}$ defined for $y \in \mathcal{C}$ by

$$y = 1^{(y)_0} {}^\frown 0 {}^\frown (y)_1.$$

Finally, conclude the statement for any Y using the perfect set property.

(b) Conclude that if X is uncountable Polish, then $\boldsymbol{\Delta}^0_\lambda(X) \supsetneq \boldsymbol{\Omega}^0_\lambda(X)$.

7. Let X, Y be topological spaces and let $\mathrm{proj}_X : X \times Y \to X$ be the projection function. Prove the following statements:

(a) proj_X is continuous and open (i.e. maps open sets to open sets).

(b) proj_X does not in general map closed sets to closed sets, even for $X = Y = R$.

Remark: We will see shortly in the course that for certain $Y = \mathcal{N}$, the projection of a closed set may not even be Borel in general.

(c) For $X = Y = R$ (or in general any σ-compact Hausdorff space), proj_X maps closed sets to σ-compact (and hence F_σ) sets.

(d) (Tube lemma) If Y is compact, then proj_X indeed maps closed sets to closed sets.

Hint: It is perhaps tempting to use sequences, but this would only work for first-countable spaces. Instead, use the open cover definition of compact and show that for closed $F \subseteq X \times Y$, every point $x \in X \setminus \mathrm{proj}_X(F)$ has an open neighborhood disjoint from $\mathrm{proj}_X(F)$. The "correct" solution should use nothing but definitions.