DESCRIPTIVE SET THEORY

HOMEWORK 7

Due on Tuesday, Mar 18

1. (Present) Let G be a Polish group (i.e. a topological group whose topology happens to be Polish) and let $H < G$ be a subgroup. Prove that H is Polish iff H is closed.

Hint: Consider H inside $\overline{H}$. What is the Baire category status (meager/non-meager/comeager) of H inside $\overline{H}$? If $H \subsetneq \overline{H}$, look at the cosets.

2. (Present) Let G be a group acting continuously on a Baire space X (i.e. each element acts as a homeomorphism of X). A set $A \subseteq X$ is called invariant if $gA = A$ for all $g \in G$. The action $G \curvearrowright X$ is called *generically ergodic* if every invariant set $A \subseteq X$ with the BP is either meager or comeager (also known as the first topological $0-1$ law). For a set $A \subseteq X$, denote by $[A]_G$ the saturation of A, namely $[A]_G = \bigcup_{g\in G} g(A)$.

(a) Prove that the following are equivalent:
  (1) $G \curvearrowright X$ is generically ergodic;
  (2) Every invariant nonempty open set is dense;
  (3) (Homogeneity) For every nonempty open sets $U, V \subseteq X$, there is $g \in G$ such that $g(U) \cap V \neq \varnothing$.

(b) Prove that if X is second countable, then the above conditions are equivalent to the existence of a dense orbit.

Hint: Prove that (1) implies there is a dense orbit by taking a countable basis $\{U_n\}_{n\in\mathbb{N}}$ and considering $\bigcap_n [U_n]_G$.

3. (Present) Show that the Kuratowski-Ulam theorem fails if A does not have the BP by constructing a non-meager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line.

Hint: Note that there are only continuum many F_σ sets, so take a transfinite enumeration $(F_\xi)_{\xi<2^{\aleph_0}}$ of all *meager* F_σ sets, and construct a sequence $(a_\xi)_{\xi<2^{\aleph_0}}$ of points in $\mathbb{R}^2$ by transfinite recursion so that for each $\xi < 2^{\aleph_0}$,

$$\{a_\lambda : \lambda \leq \xi\} \nsubseteq F_\xi,$$

and no three of the points in $\{a_\lambda : \lambda \leq \xi\}$ lie on a straight line.

Hint: Recall that in perfect Polish spaces (such as $\mathbb{R}, \mathbb{R}^2$), any non-meager subset with the BP contains a copy of the Cantor space (this is because it contains a non-meager G_δ set). Now if $A := \{a_\lambda : \lambda \leq \xi\} \subseteq F_\xi$, apply Kuratowski-Ulam to F_ξ to find $x \in \mathbb{R}$ such that $(F_\xi)_x$ is meager and the vertical line $L_x = \{(x, y) \in \mathbb{R} : y \in \mathbb{R}\}$ is disjoint from A.

4. (Present) Show that if X, Y are second countable Baire spaces, so is $X \times Y$.

5. **Definition.** A *filter* on a set X is a set $\mathcal{U} \subseteq \text{Pow}(X)$ such that

  (i) (Nontriviality) $X \in \mathcal{U}$ but $\varnothing \notin \mathcal{U}$;

(ii) (Upward closure) $A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$;

(iii) (Closure under finite intersections) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$.

A filter $\mathcal{U}$ is called an *ultrafilter* if $A \notin \mathcal{U} \Rightarrow A^c \in \mathcal{U}$ for every $A \subseteq X$. Finally, an ultrafilter is called *principal* if for some $x \in X$, $\{x\} \in \mathcal{U}$ (or, equivalently, $\mathcal{U} = \{A \subseteq X : x \in A\}$).

It is useful to think of a filter $\mathcal{U}$ as the family of all conull sets of a $\{0,1\}$-valued finitely additive measure $\mu_{\mathcal{U}}$ on a subalgebra of $\mathrm{Pow}(X)$. In other words, sets in $\mathcal{U}$ should be thought of as large sets. $\mathcal{U}$ being an ultrafilter simply means that $\mu_{\mathcal{U}}$ is defined on all of $\mathrm{Pow}(X)$; in other words, if a set is not large then it is small (i.e. there are no intermediate sets). Also, $\mathcal{U}$ being principal means that $\mu_{\mathcal{U}}$ is a Dirac measure (i.e. a point-mass at some point x).

(a) (AC) Prove that for every infinite set X, there exists a non-principal ultrafilter on X; do it by showing that every filter is contained in an ultrafilter and applying this to the filter of cofinite sets (called the Fréchet filter).

(b) (Present) Identifying $\mathrm{Pow}(\mathbb{N})$ with $\mathcal{C} = 2^{\mathbb{N}}$, view ultrafilters on $\mathbb{N}$ as subsets of $\mathcal{C}$ and show that no non-principal ultrafilter $\mathcal{U}$ has the BP (as a subset of $\mathcal{C}$).

6. (I'll present this, but you read it as it is used in the problems below.) Using the outline below, prove Pettis's theorem:

Theorem (Pettis)**.** *Let G be a topological group and $A \subseteq G$ have the BP. If A is non-meager, then $A^{-1}A$ contains an open neighborhood of the identity 1_G; in fact if $U \Vdash A$, then $U^{-1}U \subseteq A^{-1}A$.*

Step 1. Recall from the last homework that G must be Baire.

Step 2. Note that for any sets $B, C \subseteq G$,

$$B \subseteq C^{-1}C \iff \forall h \in B\ (Ch \cap C \neq \varnothing). \qquad (*)$$

Step 3. Let $U \subseteq G$ be nonempty open such that $U \Vdash A$. Fix arbitrary $g \in U$ and note that $V = g^{-1}U \subseteq U^{-1}U$ is an open neighborhood of 1_G. Thus, by $(*)$, $h \in V$, $Uh \cap U \neq \varnothing$.

Step 4. Conclude that for each $h \in V$, $Ah \cap A \neq \varnothing$, and hence, by $(*)$ again, $V \subseteq A^{-1}A$.

Step 5. Note that we have shown $g^{-1}U \subseteq A^{-1}A$ for arbitrary $g \in U$, and thus, $U^{-1}U \subseteq A^{-1}A$.

7. (Present) Let G be a Baire topological group (i.e. G is non-meager). Prove that any non-meager subgroup $H < G$ with the BP is actually clopen! In particular, if H has countable index in G, then H is clopen.

8. (Present)

(a) **Automatic continuity:** Let G, H be topological groups, where G is Baire and H is separable. Then every Baire measurable group homomorphism $\phi : G \to H$ is actually continuous!

Hint: Enough to prove continuity at 1_G, so let $U \ni 1_H$ be open and take an open neighborhood $V \ni 1_H$ such that $V^{-1}V \subseteq U$. Using the separability of H, deduce that $\phi^{-1}(hV)$ is non-meager for some $h \in H$ and apply Pettis's theorem.

(b) Conclude that if $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a Baire measurable group homomorphism, then for some $a \in \mathbb{R}$, $f(x) = ax$ for all $x \in \mathbb{R}$.

HINT: First show this for integers, then for rationals, etc.