# DESCRIPTIVE SET THEORY

HOMEWORK 6

Due on Tuesday, Mar 11

**1.** (Present) A *finite bounded game* on a set $A$ is a game similar to infinite games, but the players play at most $n$ number of steps before the winner is decided, for some fixed number $n \geq 1$ (say a million). More formally, the game is a tree $T \subseteq A^{<n}$, for some $n$, and the runs of the game are exactly the elements of the set $\text{Leaves}(T)$ of all leaves of $T$, so the payoff set is a subset $D \subseteq \text{Leaves}(T)$. Player I wins the run $s \in \text{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$. All games that appear in real life are such games, e.g. chess (counting ties as a win for Player II).

Prove the determinacy of finite bounded games.

Hint: Let's write down what it means for Player I to have a winning strategy in this game, assuming for simplicity that $n$ is even and that all of the runs of the game are of length exactly $n$:

$$\exists a_1 \forall a_2 \ldots \exists a_{n-1} \forall a_n ((a_1, \ldots, a_n) \in D).$$

What happens when you negate this statement?

**2.** (Present) A *finite game* on a set $A$ is a game similar to infinite games, but the players play only finitely many steps before the winner is decided. More formally, it is a (possibly infinite) tree $T \subseteq A^{<\mathbb{N}}$ that has no infinite branches, and the set of runs is $\text{Leaves}(T)$, so the payoff set is a subset $D \subseteq \text{Leaves}(T)$. Player I wins the run $s \in \text{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \text{Leaves}(T) \setminus D$.

(a) Prove the determinacy of finite games.

Hint: Call a position $s \in T$ determined, if from that point on, one of the players has a winning strategy. Thus, no player has a winning strategy in the beginning iff $\varnothing$ is undetermined. What can you say about extensions of undetermined positions?

(b) Conclude the determinacy of clopen infinite games (i.e. the games defined in class, where the runs are elements of $A^{\mathbb{N}}$).

**3.** (This problem is fun, think about it; it's not hard but it's tricky.) In ZF (in particular, don't use AC or ¬AD), define a game with rules $G(T, D)$ on the set $A = \text{Pow}(\mathbb{N}^{\mathbb{N}})$ (i.e. define a pruned tree $T \subseteq A^{<\mathbb{N}}$ and a set $D \subseteq A^{\mathbb{N}}$), so that ZF+¬AD implies that this game is undetermined. In other words, you have to define the tree $T$ and the payoff set $D$ without using ¬AD, but then prove that the game $G(T, D)$ is undetermined using ¬AD.

Hint: Note that besides playing subsets of $\mathbb{N}^{\mathbb{N}}$, players can also play natural numbers in the sense that $\mathbb{N} \hookrightarrow \text{Pow}(\mathbb{N}^{\mathbb{N}})$ by $n \mapsto \{(n)_{i \in \mathbb{N}}\}$.

**4.** (Present) Let $X$ be a second countable Baire space. Show that the $\sigma$-ideal $\text{MGR}(X)$ has the countable chain condition in $\text{BP}(X)$, i.e. there is no uncountable family $\mathcal{A} \subseteq \text{BP}(X)$ of non-meager sets such that for any two distinct $A, B \in \mathcal{A}$, $A \cap B$ is meager.

**5.** (Present) Let $X$ be a topological space.

(a) If $A_n \subseteq X$, then for any open $U \subseteq X$,

$$U \Vdash \bigcap_n A_n \iff \forall n (U \Vdash A_n).$$

(b) If $X$ is a Baire space, $A$ has the BP in $X$ and $U \subseteq X$ is nonempty open, then

$$U \Vdash A^c \iff \forall V \subseteq U (V \nVdash A),$$

where $V$ varies over a weak basis[1] for $X$.

(c) If $X$ is a Baire space, the sets $A_n \subseteq X$ have the BP, and $U$ is nonempty open, then

$$U \Vdash \bigcup_n A_n \iff \forall V \subseteq U \exists W \subseteq V \exists n (W \Vdash A_n).$$

where $V, W$ vary over a weak basis for $X$.

**6.** (Present) Let $G$ be a topological group, i.e. a group with a topology on it so that group multiplication $(x, y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Prove that $G$ is Baire iff $G$ is non-meager.

**7.** (Present) Let $X$ be a topological space and $A \subseteq X$.

(a) Show that $U(A)$ is regular open, i.e. it is equal to the interior of its closure.

(b) If moreover $X$ is a Baire space and $A$ has the BP, then $U(A)$ is the unique regular open set $U$ with $A =^* U$.

[1] A *weak basis* for a topological space $X$ is a collection $\mathcal{V}$ of nonempty open sets such that every nonempty open set $U \subseteq X$ contains at least one $V \in \mathcal{V}$.