DESCRIPTIVE SET THEORY

HOMEWORK 5

Due on Tuesday, Mar 4

1. Show that the perfect kernel of a Polish space X is the largest perfect subset of X, i.e. it contains all other perfect subsets.

2. (Present) Let X be separable metrizable. Show that

$$\mathcal{K}_p(X) = \{K \in K(X) : K \text{ is perfect}\}$$

is a G_δ set in $K(X)$. In particular, if X is Polish, then so is $\mathcal{K}_p(X)$.

3. (Present)

(a) Let X be a Polish space. Show that if $K \subseteq X$ is countable and compact, then its Cantor-Bendixson rank $|K|_{\mathrm{CB}}$ is not a limit ordinal.

(b) For each non-limit ordinal $\alpha < \omega_1$, construct a countable compact subset K_α of $\mathcal{C}$, whose Cantor-Bendixson rank is exactly α.

4. (Present) Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree.

(a) Suppose T is pruned. Find a condition (on the nodes of the tree T) such that T satisfies it if and only if $[T]$ is perfect (as a subset of $\mathcal{N}$).

(b) Define a Cantor-Bendixson derivative T' of T, as well as the iterated derivatives $(T^\alpha)_{\alpha \in \mathrm{ON}}$, such that $[T^\infty]$ is the perfect kernel of $[T]$, i.e. $[T^\infty] = [T]^\infty$.

Remark: The statement of this problem is somewhat vague and informal, but understanding it is part of the challenge.

5. Let X be a second countable zero-dimensional space.

(a) Prove Kuratowski's reduction property: If $A, B \subseteq X$ are open, there are open $A^* \subseteq A, B^* \subseteq B$ with $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \varnothing$.

Hint: Write A and B as countable unions of clopen sets: $A = \bigcup_n A_n$, $B = \bigcup_n B_n$. Put those points x of A in A^* that are covered by A_n no later than by B_n, i.e. if n is the smallest number such that $x \in A_n \cup B_n$, then $x \in A_n$.

(b) Conclude the following separation property: For any disjoint closed sets $A, B \subseteq X$, there is a clopen set C separating A and B, i.e. $A \subseteq C$ and $B \cap C = \varnothing$.

6. (Present)

(a) Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and prove that there is a Luzin scheme $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ that has vanishing diameter and satisfies the following:

(i) $A_\varnothing = X$;

(ii) A_s is nonempty clopen;

(iii) $A_s = \bigcup_{i \in \mathbb{N}} A_{s^\frown i}$.

HINT: Assuming A_s is defined, cover it by countably many clopen sets of diameter at most $\delta < 1/n$, and choose the δ small enough so that any such cover is necessarily infinite.

(b) Derive the Alexandrov-Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis of (a).

(c) Let $X \subseteq \mathbb{R}$ be G_δ and such that $X, \mathbb{R} \setminus X$ are dense in $\mathbb{R}$. Show that X is homeomorphic to $\mathcal{N}$. Prove that the same fact also holds when $\mathbb{R}$ is replaced by a zero-dimensional nonempty Polish space. Show that it may fail if $\mathbb{R}$ is replaced by $\mathbb{R}^2$.

7. The following steps outline a proof of the Baire category theorem for locally compact Hausdorff spaces.

(1) Show that compact Hausdorff spaces are normal.

(2) Using part (1), prove that in locally compact Hausdorff space X, for every nonempty open set U and every point $x \in U$, there is a nonempty open precompact[1] $V \subseteq U$ with $\overline{V} \subseteq U$.

(3) Prove that locally compact Hausdorff spaces are Baire.

8. (Present)

(a) Show that category notions are preserved under continuous open preimages; more precisely, for a continuous open map $f : X \to Y$ between topological spaces X, Y, and a set $B \subseteq Y$, if B is nowhere dense (resp. meager, dense, comeager), then so is $f^{-1}(B)$. In particular, this is true about projections.

(b) Show that part (a) is false if preimages are replaced by images; i.e. construct an example of a continuous open map $f : X \to Y$ and a nowhere dense set $A \subseteq X$ such that $f(A)$ is not nowhere dense.

9. (I will present this one, but you read/think about it.) Recall that $C([0,1])$ is a Polish space with the uniform metric. Show that the generic element of $C([0,1])$ is nowhere differentiable following the outline below.

(1) Prove that given $m \in \mathbb{N}$, any function $f \in C([0,1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0,1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geq m$.

(2) For each $n \geq 1$, let E_n be the set of all functions $f \in C([0,1])$, for which there is $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$. Show that E_n is nowhere dense using the fact that if g is as in (1) with $m = 2n$, then some open neighborhood of g is disjoint from E_n.

10. (Present) Let X be a perfect Polish space and show that a generic compact subset of X is perfect, i.e. show that the set $\mathcal{K}_p(X)$ is comeager in $K(X)$ (see Problem 2).

[1] Precompact sets are those whose closure is compact.