DESCRIPTIVE SET THEORY

HOMEWORK 4

Due on Tuesday, Feb 25

1. Following the outline below, prove the equivalence of the three definitions of compactness for metric spaces listed in Proposition 3.4.

(1) ⇒ (2): For a sequence $(x_n)_n$, let K_m be the closure of the tail $\{x_n\}_{n \geq m}$ of the sequence and use the intersection-of-closed sets version of the definition of compactness.

(2) ⇒ (3): For total boundedness, fix an $\epsilon > 0$ and start constructing an ϵ-net F by adding elements to your F that are not yet covered by $B(F, \epsilon)$. For completeness, note that if a subsequence of a Cauchy sequence converges, then so does the entire sequence.

(3) ⇒ (2): Let $(x_n)_n$ be a sequence in X. Think of the x_n-s as pigeons and a finite ϵ-net as holes.

(2) and (3) ⇒ (1): Somewhat trickier, look up Theorem 0.25 in Folland's "Real Analysis".

2. (I will present this one, but you think about it.) Let X be a compact metric space and Y be a separable complete metric space. Let $C(X, Y)$ be the space of continuous functions from X to Y equipped with the uniform metric, i.e. for $f, g \in C(X, Y)$,

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), f(y)).$$

Prove that $C(X, Y)$ is a separable complete metric space, hence Polish.

Hint 1: Proving separability is tricky, so you may want to first prove it for $X = [0, 1]$ and $Y = \mathbb{R}$. In the general case (to prove separability), note that by uniform continuity,

$$C(X, Y) = \bigcup_n A_{n,m},$$

for every $n \in \mathbb{N}$, where

$$A_{n,m} = \{f \in C(X, Y) : \forall x, y \in X (d_X(x, y) < 1/n \Rightarrow d_Y(f(x), f(y)) < 1/m)\}.$$

Realize that it is enough to show that for any $n, m \in \mathbb{N}$, there is a countable $B_{n,m} \subseteq A_{n,m}$ such that for any $f \in A_{n,m}$ there is $g \in B_{n,m}$ with $d_u(f, g) < 3/m$. Now fix n, m and try to construct $B_{n,m}$; when doing so, don't try to *define* each function in $B_{n,m}$ by hand as you would maybe do in the case $X = [0, 1]$; instead, carefully *pick* them out of functions in $A_{n,m}$.

Hint 2: This is Theorem 4.19 in Kechris's "Classical Descriptive Set Theory".

3. (Present) Show that Hausdorff metric on $K(X)$ is compatible with the Vietoris topology.

4. (Present) Let (X, d) be a metric with $d \leq 1$. For $(K_n)_n \subseteq K(X) \setminus \{\varnothing\}$ and nonempty $K \in K(X)$:

(a) $\delta(K, K_n) \to 0 \Rightarrow K \subseteq \underline{\mathrm{T}\lim}_n K_n$;

(b) $\delta(K_n, K) \to 0 \Rightarrow K \supseteq \overline{\mathrm{T}\lim}_n K_n$.

In particular, $d_H(K_n, K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. Show that the converse may fail.

5. Let (X, d) be a compact metric with $d \leq 1$. For sequence $(K_n)_n \subseteq K(X) \setminus \{\varnothing\}$, show the following:

(a) if $\underline{\mathrm{T}\lim}_n K_n \neq \varnothing$ then $\delta(\underline{\mathrm{T}\lim}_n K_n, K_m) \to 0$ as $m \to \infty$;

(b) $\delta(K_m, \overline{\mathrm{T}\lim}_n K_n) \to 0$ as $m \to \infty$.

So if $K = \mathrm{T}\lim_n K_n$ exists, $d_H(K_n, K) \to 0$.

6. Let (X, d) be a metric space with $d \leq 1$. Then $x \mapsto \{x\}$ is an isometric embedding of X into $K(X)$.

7. (Present) Let (X, d) be a metric space with $d \leq 1$ and assume $K_n \to K$. Then any sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in K_n$ has a subsequence converging to a point in K.

8. Let X be metrizable.

(a) (Present) The relation "$x \in K$" is closed, i.e. $\{(x, K) : x \in K\}$ is closed in $X \times K(X)$.

(b) The relation "$K \subseteq L$" is closed, i.e. $\{(K, L) : K \subseteq L\}$ is closed in $K(X)^2$.

(c) (Present) The relation "$K \cap L \neq \varnothing$" is closed, i.e. $\{(K, L) : K \cap L \neq \varnothing\}$ is closed in $K(X)^2$.

(d) (Present) The map $(K, L) \mapsto K \cup L$ from $K(X)^2$ to $K(X)$ is continuous.

(e) If Y is metrizable, then the map $(K, L) \mapsto K \times L$ from $K(X) \times K(Y)$ into $K(X \times Y)$ is continuous.

(f) (Present) Find a compact X for which the map $(K, L) \mapsto K \cap L$ from $K(X)^2$ to $K(X)$ is not continuous.

9. (Present) Let X be a topological space.

(a) If X is nonempty perfect, then so is $K(X) \setminus \{\varnothing\}$.

(b) If X is compact metrizable, then $C(X)$ is perfect, where $C(X) = C(X, \mathbb{R})$.

10. (Present) Show that any nonempty perfect compact Hausdorff space X has cardinality at least continuum by constructing an injection from the Cantor space into X.

Hint: Mimic the proof for Polish spaces.

11. (Present) Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. Conclude that $\mathbb{Q}$ is not Polish in the relative topology of $\mathbb{R}$.