DESCRIPTIVE SET THEORY

HOMEWORK 3

Due on Tuesday, Feb 18

1. (Present) Prove that any separable metric space has cardinality at most continuum.

Remark: This is true more generally for first-countable separable Hausdorff topological spaces, but false for general separable Hausdorff topological spaces (try to construct a counter-example).

2. (Present)

(a) Show that a metric space X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n \in \mathbb{N}}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n \in \mathbb{N}} B_n$ is a singleton).

(b) Show that the requirement in (a) that $\operatorname{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of closed **balls** with $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$.

Hint: Use $\mathbb{N}$ as the underlying set for your metric space.

3. (Present) By definition, the class of G_δ sets is closed under countable intersections. Show that it is also closed under finite unions. Equivalently, the class of F_σ sets is closed under finite intersections.

Hint: Think in terms of quantifiers $\forall$ and $\exists$ rather than intersections and unions; for example, if $A = \bigcap_n U_n$, then $x \in A \iff \forall n (x \in A_n)$.

4. (Present)

(a) Show that the Cantor set (with relative topology of $\mathbb{R}$) is homeomorphic to the Cantor space.

(b) Show that the Baire space $\mathcal{N}$ is homeomorphic to a G_δ subset of the Cantor space $\mathcal{C}$.

(c) Show that the set of irrationals (with the relative topology of $\mathbb{R}$) is homeomorphic to the Baire space.

Hint: Use the continued fraction expansion.

5. (Present) Let $T \subseteq A^{<\mathbb{N}}$ be a tree and suppose it is finitely branching. Prove that $[T]$ is compact.

6. (Present) Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T doesn't have an infinite branch.

7. Let S, T be trees on sets A, B, respectively. Prove the following using the outline below: If $f : G \to [T]$ is continuous, where $G \subseteq [S]$ is G_δ, then there is monotone $\phi : S \to T$ with $f = \phi^*$.

1. To understand the basic idea, first prove the statement assuming that $G = [S]$. In this case, let $\phi(s)$ be the longest $u \in T$ such that $|u| \le |s|$ and $N_u \supseteq f(N_s)$.
2. Now assuming that $G = \bigcap_{n \in \mathbb{N}} U_n$, where $(U_n)_{n \in \mathbb{N}}$ is a decreasing sequence of open sets in $[S]$ with $U_0 = [S]$, modify the above definition to bound $|u|$ with $k(s)$ instead of $|s|$, where $k(s)$ is equal to the largest number $k \le |s|$ such that $U_k \supseteq N_s \cap [S]$.