DESCRIPTIVE SET THEORY

HOMEWORK 2

Due on Tuesday, Feb 11

1. (Present) Prove that Zermelo's Theorem implies AC.

Caution: It is easy to accidentally use AC in your proof. Make sure you don't.

2. (Present) Prove that AC implies Zorn's Lemma.

Outline: Let $(A, <)$ be as in the statement of Zorn's Lemma and assume for contradiction that no element is maximal. Then for every $a \in A$, the set $\operatorname{succ}(a) := \{b \in A : a < b\}$ is nonempty. Thus, by AC, there is a function $f : A \to A$ mapping each a to an element in $\operatorname{succ}(a)$. Similarly, for every chain $B \subseteq A$, the set $U(B) := \{a \in A : \forall b \in B (b < a)\}$ is nonempty. Hence, denoting the set of all chains in A (including the empty set) by $\operatorname{Chains}(A)$, AC provides a function $g : \operatorname{Chains}(A) \to A$ mapping each chain B to an element in $U(B)$. Obtain a contradiction by defining an injection of $\chi(A)$ into A using transfinite induction: use f to define the successor cases and use g to define the 0 and the limit cases.

Remark: Alternatively, by the same transfinite induction, one can obtain an injection of ON into A, contradicting ON being a proper class.

3. (Present) Prove that Zorn's Lemma implies Zermelo's Theorem.

Hint: For a set A, consider $(\mathrm{WO}(A), \prec)$, where $\mathrm{WO}(A)$ is as in the proof of Hartogs' theorem in the notes.

4. Prove the following version of Hartogs' theorem that doesn't use Replacement axiom:

Theorem (Hartogs without Replacement)**.** *For every set A, there is a well-ordered set $(H(A), <_{H(A)})$ such that $H(A) \nleq A$ and $(H(A), <_{H(A)})$ is $\preceq$-least such well-ordering, i.e. if $(B, <_B)$ is another well-ordering with the property that $B \nleq A$, then $(H(A), <_{H(A)}) \preceq (B, <_B)$.*

Outline: Denote by $H(A)$ the quotient of $\mathrm{WO}(A)$ by the equivalence relation $\simeq$, i.e. $H(A) = \mathrm{WO}(A)/\simeq$. For $(B, <_B) \in \mathrm{WO}(A)$, let $[(B, <_B)]$ denote the $\simeq$-equivalence class of $(B, <_B)$ in $H(A)$. Define an ordering $<_{H(A)}$ on $H(A)$ as follows:

$$[(B, <_B)] <_{H(A)} [(C, <_C)] \iff (B, <_B) \prec (C, <_C),$$

for $[(B, <_B)], [(C, <_C)] \in H(A)$. Show that $<_{H(A)}$ is actually a well-ordering and verify that $(H(A), <_{H(A)})$ satisfies the conclusion of the theorem.

5. (Present)

(a) Prove without using AC that for a cardinal $\kappa \geq \omega$, $|\kappa \times \kappa| = \kappa$.

Hint: Define a well-ordering $<_2$ of $\kappa \times \kappa$ (using the well-ordering of κ) such that the cardinality of every proper initial segment of $(\kappa \times \kappa, <_2)$ is less than κ (think of how you would do it for $\kappa = \omega$). Conclude that $\mathrm{tp}(\kappa \times \kappa, <_2) \leq \kappa$.

(b) (AC) Conclude that if A_α are sets of cardinality at most κ, for $\alpha < \kappa$, then $|\bigcup_{\alpha<\kappa} A_\alpha| \leq \kappa$. Pinpoint exactly where you use AC.

(c) (AC) Show that for any infinite cardinal κ, κ^+ is regular. In particular, ω_1 is regular.

6. An *open interval* in ω_1 is a set of the form $(\alpha, \beta) := \{\gamma \in \omega_1 : \alpha < \gamma < \beta\}$ or $[0, \alpha) := \alpha$, for some $\alpha < \beta < \omega_1$. The topology generated by open intervals is naturally called the *open interval topology*.

(AC) Prove that the open interval topology on ω_1 is sequentially compact (i.e. every sequence has a convergent subsequence), but not compact (in the sense of open covers).

7. (Present) Let X be a second-countable topological space.

(a) Show that X has at most continuum many open subsets.

(b) Let α, β, γ denote ordinals. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

Hint: Use the same idea as in the proof of (a).

(c) Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ open subsets of X eventually stabilizes, i.e. there is $\gamma < \omega_1$ such that for all $\alpha < \omega_1$ with $\alpha \geq \gamma$, we have $U_\alpha = U_\gamma$.

Hint: Use the regularity of ω_1.

(d) Conclude that parts (a), (b) and (c) are also true for closed sets.