DESCRIPTIVE SET THEORY

HOMEWORK 13

Due on Tuesday, May 6

Mandatory problems.

1. (Prisoners and hats) The purpose of this problem is to illustrate the non-smoothness of $\mathbb{E}_0$, more particularly, how having a selector for $\mathbb{E}_0$ (provided by AC) can cause crazy things.

ω-many prisoners were sentenced to death, but they could get out under one condition: on the day of the execution they will be lined up, i.e. enumerated $(p_n)_{n\in\omega}$, so that everybody can see everybody else (but not themselves). Each of the prisoners will have a red or blue hat put on him, but he won't be told which color it is (although he can see the colors of other prisoners' hats). On command, all the prisoners (at once) make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they go home free; otherwise ☠. The good news is that the prisoners thought of a plan the day before the execution, and indeed, all but finitely many prisoners guessed correctly the next day, so everyone was saved. How did they do it?

2. For a Borel equivalence relation E, show that if there is universally measurable reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$, then E is smooth (i.e. there is a *Borel* reduction of E to $\mathrm{Id}(2^{\mathbb{N}})$).

Hint: It's ok to use big theorems.

3. Let $\mathcal{G}$ be a Borel graph on a Polish space $(X,\mathcal{T})$. For $A \subseteq X$, let

$$N_{\mathcal{G}}(A) = \{x \in X : \exists y \in A((x,y) \in \mathcal{G})\}$$

denote the set of $\mathcal{G}$-neighbors of vertices in A. If $A = \{x\}$, we just write $N_{\mathcal{G}}(x)$. Call $\mathcal{G}$ locally countable (resp. finite) if for every $x \in X$, $N(x)$ if countable (resp. finite).

(a) Suppose $\mathcal{G}$ is such that for every Borel $A \subseteq X$, $N_{\mathcal{G}}(A)$ is Borel, and prove that $\chi_{\mathcal{B}}(\mathcal{G}) \leq \aleph_0$ iff there is a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that for every $x \in X$, $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$.

Hint: For $\Leftarrow$, use the fact that $x \notin \overline{N_{\mathcal{G}}(x)}^{\mathcal{T}_0}$ is witnessed by a basic open set U_n (in $\mathcal{T}_0$).

(b) Conclude if $\mathcal{G}$ is locally finite, then $\chi_{\mathcal{B}}(\mathcal{G}) \leq \aleph_0$.

(c) Prove the Borel version of Vising's theorem, namely that if the maximum degree of $\mathcal{G}$ is $\leq d$ (i.e. $|N_{\mathcal{G}}(x)| \leq d$ for every $x \in X$), then $\chi_{\mathcal{B}}(\mathcal{G}) \leq d+1$.

Hint: First prove this theorem for a finite graph so you can see the algorithm. Now for our Borel $\mathcal{G}$, use part (b) to partition $X = \bigsqcup_n A_n$ into $\mathcal{G}$-independent Borel sets and start running your coloring algorithm on A_0 until you've colored all the vertices connected to A_0. Then do the same thing with the left over vertices of A_1, then with A_2, and so on.

(d) Conclude that the Borel chromatic number of the graph induced by an irrational rotation of the unit circle is 3.

4. Let $S \subseteq 2^{<\mathbb{N}}$.

(a) If S contains at most one element of each length, then $\mathcal{G}_S$ is acyclic[1].

Hint: Suppose there is a cycle (with no repeating vertex) and consider the longest $s \in S$ associated with its edges.

[1] Here we treat $\mathcal{G}_S$ as an undirected graph, i.e. we consider edges (x,y) and (y,x) to be the same.

(b) If S contains at least one element of each length, then $E_{\mathcal{G}_S} = \mathbb{E}_0$.

HINT: For each $n \in \mathbb{N}$, show by induction on n that for each $s, t \in 2^n$ and $x \in 2^{\mathbb{N}}$, there is a path in $\mathcal{G}_S$ from $s^\frown x$ to $t^\frown x$, i.e. $s^\frown x$ can be transformed to $t^\frown x$ by a series of appropriate bit flips.

Not mandatory yet interesting problems.
These problems are not hard, and I'd be happy to see you solutions (if any) in the problem session.

5. Show that there exists a *universal analytic equivalence relation*, i.e. an analytic equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_{\boldsymbol{\Sigma}}$.

HINT: Take a $\mathcal{C}$-universal set $U \subseteq \mathcal{C} \times \mathcal{N}^2$ for $\boldsymbol{\Sigma}^1_1(\mathcal{N}^2)$ and let $\tilde{U}$ be obtained from U by replacing the fibers U_x, $x \in \mathcal{C}$, with their symmetric and transitive closures, so that each fiber $\tilde{U}_x$ is an equivalence relation on $\mathcal{N}$. Now define an appropriate equivalence relation $\mathbb{E}_{\boldsymbol{\Sigma}}$ on $\mathcal{C} \times \mathcal{N}$.

6. The goal of this problem is to show that there is a *universal countable Borel equivalence relation*, i.e. a countable Borel equivalence relation $\mathbb{E}_\infty$ such that any other such equivalence relation is Borel reducible to $\mathbb{E}_\infty$.

(a) Let $G \curvearrowright X$ be a Borel action of a countable group on a Polish space X. Show that there is a Borel equivariant[2] map $f : X \to 2^G$, where $G \curvearrowright 2^G$ by shift as follows: $g \cdot y(h) = y(g^{-1}h)$, for $g, h \in G$, $y \in (2^{\mathbb{N}})^G$. In particular, f is a Borel reduction of the orbit equivalence relations.

HINT: Let $(U_n)_n$ be a countable basis for X. For $x \in X$, to define $f(x)$, for each $g \in G$ and $n \in \mathbb{N}$, record whether or not $g \cdot x \in U_n$; this gives an element of $(2^{\mathbb{N}})^G$.

(b) Letting $\mathbb{F}_\omega$ be the free group on ω-many generators and G be any countable group, define a Borel reduction $\rho : (2^{\mathbb{N}})^G \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ of the orbit equivalence relation E_G to the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift actions of G and $\mathbb{F}_\omega$, respectively.

HINT: Every countable group is a homomorphic image of $\mathbb{F}_\omega$.

(c) Using the Feldman-Moore theorem stated below, conclude that the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift action of $\mathbb{F}_\omega$ on $(2^{\mathbb{N}})^{\mathbb{F}_\omega}$ is a universal countable Borel equivalence relation.

Theorem (Feldman-Moore). *Every countable Borel equivalence relation on a Polish space is induced as the orbit equivalence relation of a Borel action of a countable group.*

[2]A map is called equivariant if it commutes with the action, i.e. $g \cdot f(x) = f(g \cdot x)$, for $x \in X$.