DESCRIPTIVE SET THEORY

HOMEWORK 12

Due on Tuesday, Apr 29

All problems below are to be presented.

1. Show that a countable topological group is Polish if and only if it is discrete.

2. Let $X_0 = \{x \in 2^{\mathbb{N}} : \forall^\infty n \; x(n) = 0\}$, $X_1 = \{x \in 2^{\mathbb{N}} : \forall^\infty n \; x(n) = 1\}$, and put $X = 2^{\mathbb{N}} \setminus (X_0 \cup X_1)$. Note that X_0 and X_1 are $\mathbb{E}_0$-classes, so all we did is throwing away from $2^{\mathbb{N}}$ two $\mathbb{E}_0$-classes. Define a continuous action of $\mathbb{Z}$ on X so that the induced orbit equivalence relation $E_{\mathbb{Z}}$ is exactly $\mathbb{E}_0 \restriction_X$.

3. Show that $\mathbb{E}_v \sim_B \mathbb{E}_0$ by proving that $\mathbb{E}_v \sqsubseteq_B \mathbb{E}_0(\mathbb{N}) \sqsubseteq_c \mathbb{E}_0$ and $\mathbb{E}_0 \sqsubseteq_c \mathbb{E}_v$.

Hint: Use that each $x \in \mathbb{R}$ can be uniquely written as

$$x = \frac{a_1}{1!} + \frac{a_2}{2!} + \cdots + \frac{a_n}{n!} + \cdots,$$

where $a_1 = \lfloor x \rfloor$, for each $n \geq 2$, $a_n \in \{0, 1, \ldots, n-1\}$ and $\exists^\infty n (a_n \neq n-1)$; the latter condition is to ensure uniqueness.

4. Let E be an equivalence relation on a Polish space X. Prove that $\mathrm{id}(2^{\mathbb{N}}) \leq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_B E$ iff $\mathrm{id}(2^{\mathbb{N}}) \sqsubseteq_c E$.

5. Fill in the details in the proof of Mycielski's theorem; namely, given a meager equivalence relation E on a Polish space X, write $E = \bigcup_n F_n$, where F_n are increasing and nowhere dense, and construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}} \subseteq X$ of vanishing diameter (with respect to a fixed complete metric d for X) with the following properties:

(i) U_s is nonempty open, for each $s \in 2^{<\mathbb{N}}$;

(ii) $\overline{U_{s^\frown i}} \subseteq U_s$, for each $s \in 2^{<\mathbb{N}}$, $i \in \{0, 1\}$;

(iii) $(U_s \times U_t) \cap F_n = \varnothing$, for all distinct $s, t \in 2^n$ and $n \in \mathbb{N}$.

6. Let $(X, \mathcal{T})$ be a Polish space and let E be an equivalence relation on X. E is called *smooth* if $E \leq_B \mathrm{Id}(\mathbb{R})$.[1] For a family $\mathcal{F}$ of subsets of X, we say that $\mathcal{F}$ *generates* E if

$$xEy \iff \forall A \in \mathcal{F} (x \in A \Leftrightarrow y \in A).$$

Prove that the following are equivalent:

(1) E is smooth;

(2) There is a Polish topology $\mathcal{T}_E \supseteq \mathcal{T}$ on X (and hence automatically $\mathcal{B}(\mathcal{T}_E) = \mathcal{B}(\mathcal{T})$) such that E is closed in $(X^2, \mathcal{T}_E^2)$.

Caution: It is easy to make E closed in X^2 by refining the topology of X^2, but here we have to refine the topology of X so that E becomes closed in X^2.

(3) E is generated by a countable Borel family $\mathcal{F} \subseteq \mathcal{B}(\mathcal{T})$.

Hint: For (1)⇒(2), consider a Borel function witnessing the smoothness of E and make it continuous. For (2)⇒(3), assuming that E is closed, write $X^2 \setminus E = \bigcup_n U_n \times V_n$, with U_n, V_n basic open, and note that the saturations $[U_n]_E$ and $[V_n]_E$ are disjoint analytic sets; separate them by an invariant Borel set.

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[1] Note that by the Borel isomorphism theorem, $\mathbb{R}$ can be replaced with any other uncountable Polish space.