# DESCRIPTIVE SET THEORY

HOMEWORK 11

Due on Tuesday, Apr 22

**All problems below are to be presented.**

**1.** For a topological space $X$, show that $\mathrm{BP}(X)$ admits envelops: for a given $A \subseteq X$, first find a $\mathrm{BP}(X)$-envelop for it in terms of $U(\cdot)$, then write down explicitly what the set is.

**2.** Let $X$ be a Polish space and let $\mathbf{C}(X)$ denote the smallest $\sigma$-algebra on $X$ containing $\mathcal{B}(X)$ and closed under the operation $\mathcal{A}$.

(a) Show that $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X) \subseteq \mathbf{C}(X)$.
Hint: For $\sigma(\mathbf{\Sigma}^1_1(X)) \subseteq \mathcal{A}\mathbf{\Pi}^1_1(X)$, it is enough to show that $\mathcal{A}\mathbf{\Pi}^1_1(X)$ is closed under countable unions and countable intersections. For countable unions, use the natural bijection $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}^{<\mathbb{N}} \smallsetminus \{\varnothing\}$ given by $(n, s) \mapsto n^\frown s$. For countable intersections, use the usual diagonal (snake-like) bijection $\mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$ to monotonically encode finite sequences of elements of $\mathbb{N}^{<\mathbb{N}}$ into single elements of $\mathbb{N}^{<\mathbb{N}}$.

(b) For each uncountable Polish space $Y$ show that there is a $Y$-universal set for $\mathcal{A}\mathbf{\Pi}^1_1(X)$.
Hint: Enough to prove for $Y = \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}}$ (why?). Start with a $\mathcal{N}$-universal set $F \subseteq \mathcal{N} \times X$ for $\mathbf{\Pi}^1_1(X)$ and for each $s \in \mathbb{N}^{<\mathbb{N}}$, consider the set $P_s \subseteq \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$ defined as follows: for $(y, x) \in \mathcal{N}^{\mathbb{N}^{<\mathbb{N}}} \times X$, put $(y, x) \in P_s :\Leftrightarrow (y(s), x) \in F$.

(c) Conclude that for uncountable $X$, $\sigma(\mathbf{\Sigma}^1_1(X)) \subsetneq \mathcal{A}\mathbf{\Pi}^1_1(X) \subsetneq \mathbf{C}(X)$.

**3.** (Fun problem) Prove directly (without using Wadge's theorem or lemma) that any countable dense $Q \subseteq 2^{\mathbb{N}}$ is $\mathbf{\Sigma}^0_2$-complete, by showing that player II has a winning strategy in the Wadge game $G_W(A, Q)$ for any $A \in \mathbf{\Sigma}^0_2(\mathcal{N})$.

**4.** For a property $P \subseteq \mathbb{N}$ of natural numbers, we use the following abbreviations:

$$\forall^\infty n P(n) \;:\Leftrightarrow\; \{n \in \mathbb{N} : P(n)\} \text{ is cofinite } \Leftrightarrow \text{ for large enough } n,\ P(n) \text{ holds}$$
$$\exists^\infty n P(n) \;:\Leftrightarrow\; \{n \in \mathbb{N} : P(n)\} \text{ is infinite } \Leftrightarrow \text{ for arbitrarily large } n,\ P(n) \text{ holds}$$

Show that the set $Q_2 = \{x \in 2^{\mathbb{N}} : \forall^\infty n (x(n) = 0)\}$ is $\mathbf{\Sigma}^0_2$-complete and conclude that the set $N_2 = \{x \in 2^{\mathbb{N}} : \exists^\infty n (x(n) = 0)\}$ is $\mathbf{\Pi}^0_2$-complete.

**5.** Show that the following sets are $\mathbf{\Pi}^0_3$-complete:

(a) $P_3 = \{x \in 2^{\mathbb{N}\times\mathbb{N}} : \forall n \forall^\infty m (x(n, m) = 0)\}$,
Hint: Use $Q_2$ from the previous problem.

(b) $C_3 = \{x \in \mathbb{N}^{\mathbb{N}} : \lim_n x(n) = \infty\}$.
Hint: Reduce $P_3$ to $C_3$.

**6.** Each binary relation on $\mathbb{N}$ is an element of $\mathrm{Pow}(\mathbb{N}^2)$, which we may identify with $2^{\mathbb{N}^2}$. Thus, we can define

$$\mathrm{LO} = \{x \in 2^{\mathbb{N}^2} : x \text{ is a linear ordering}\}$$
$$\mathrm{WO} = \{x \in 2^{\mathbb{N}^2} : x \text{ is a well-ordering}\}.$$

(a) Show that LO is a closed subset of $2^{\mathbb{N}^2}$ and that WO is co-analytic.

(b) Prove that WO is actually $\mathbf{\Pi}^1_1$-complete.
Hint: Define an appropriate ordering on a tree to show that $\mathrm{WF} \leq_W \mathrm{WO}$, where $\mathrm{WF} = \mathrm{Tr} \smallsetminus \mathrm{IF}$.