DESCRIPTIVE SET THEORY

HOMEWORK 10

Due on Tuesday, Apr 15

All problems below are to be presented.

1. Let X be set and let $\mathcal{T}, \mathcal{T}'$ be Polish topologies on X such that $T \subseteq \mathcal{B}(T')$ (for example, this would hold if $T \subseteq T'$). Show that $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}')$.

2. Prove the following characterization of Borel sets: A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$. (Even though I mentioned this statement in class, it is still instructive to go over the argument once again by yourself.)

3. Prove that any standard Borel space $(X, \mathcal{S})$ admits a Borel linear ordering, i.e. there is a linear ordering $<$ of X such that $<$ is Borel as a subset of X^2 (with respect to the product σ-algebra).

4. Let X be Polish and consider the coding map $c : F(X) \to 2^{\mathbb{N}}$ defined by $F \mapsto$ the characteristic function of $\{n \in \mathbb{N} : F \cap U_n \neq \varnothing\}$. Prove that for $x \in 2^{\mathbb{N}}$, $x \in c(F(X))$ if and only if

$$\forall U_n \subseteq U_m [x(n) = 1 \to x(m) = 1]$$

and

$$\forall U_n \forall \epsilon \in \mathbb{Q}^+ [x(n) = 1 \to (\exists m, \text{ with } \overline{U_m} \subseteq U_n \text{ and } \operatorname{diam}(U_m) < \epsilon)\ x(m) = 1].$$

Conclude that $c(F(X))$ is a G_δ subset of $2^{\mathbb{N}}$ and hence the Effros space $F(X)$ is standard Borel.

5. Let X be a Polish space. A function $s : F(X) \to X$ is called a selector if $s(F) \in F$ for every nonempty $F \in F(X)$. The goal of this problem is to show that for every Polish space X, the Effros Borel space $F(X)$ admits a Borel selector.

(a) Show that $F(\mathcal{N})$ admits a Borel selector.

(b) Show that there is a continuous open surjection $g : \mathcal{N} \to X$ by constructing a scheme $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of open sets such that $U_\varnothing = X$, $\overline{U}_{s^\frown i} \subseteq U_s$, $U_s = \bigcup_i U_{s^\frown i}$ and $\operatorname{diam}(U_s) < 2^{-|s|}$. Caution: We don't require $U_{s^\frown i} \cap U_{s^\frown j} = \varnothing$ for $i \neq j$ (which makes your life easy), so the associated map g may not be injective.

(c) Prove that the map $f : F(X) \to F(\mathcal{N})$ defined by $F \mapsto g^{-1}(F)$ is Borel.

(d) Conclude that $F(X)$ admits a Borel selector.

6. Let X, Y be Polish spaces and let $f : X \to Y$ be a continuous function such that $f(X)$ is uncountable. Put

$$\mathcal{K}_f(X) = \{K \in K(X) : f\restriction_K \text{ is injective}\},$$

and note that for $K \in K(X)$,

$$K \in \mathcal{K}_f(X) \iff \forall U_1, U_2 \in \mathcal{U} \text{ with } \overline{U_1} \cap \overline{U_2} = \varnothing [f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \varnothing].$$

Next, show that for fixed $U_1, U_2 \in \mathcal{U}$ with $\overline{U_1} \cap \overline{U_2} = \varnothing$ the set

$$\mathcal{V} = \{K \in K(X) : f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \varnothing\}$$

is open in $K(X)$, and hence $\mathcal{K}_f(X)$ is G_δ.

7. Let X be a Polish space, $F \subseteq X \times \mathcal{N}$ and $A = \text{proj}_X(F)$. Show that if Player II has a winning strategy in the unfolded Banach-Mazur game $G^{**}(F, X)$, then A is meager.