SATURATION OF ULTRAPRODUCTS

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Throughout, let κ be an infinite cardinal. A function $f : \mathscr{P}_{\text{fin}}(\kappa) \to \mathscr{P}(\kappa)$ is called *monotone* if for every $a, b \in \mathscr{P}_{\text{fin}}(\kappa)$, $a \subseteq b \Rightarrow f(a) \supseteq f(b)$. We would call f a *homomorphism* if $f(a \cup b) = f(a) \cap f(b)$.

Definition 1 (Keisler, [Kei10, 10.1]). An ultrafilter $\mathcal{U}$ on κ is said to be *good* if for any monotone function $f : \mathscr{P}_{\text{fin}}(\kappa) \to \mathcal{U}$, there is a homomorphism $g : \mathscr{P}_{\text{fin}}(\kappa) \to \mathcal{U}$ with $g(a) \subseteq f(a)$ for every $a \in \mathscr{P}_{\text{fin}}(\kappa)$.

Proposition 2. *Any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is good.*

Proof. Given a monotone function $f : \mathscr{P}_{\text{fin}}(\mathbb{N}) \to \mathcal{U}$, define $g : \mathscr{P}_{\text{fin}}(\mathbb{N}) \to \mathcal{U}$ by $g(a) := f([0, \max(a)])$. Because f is monotone, we have $g(a) \subseteq f(a)$. Moreover, g is a homomorphism because for any $a, b \in \mathscr{P}_{\text{fin}}(\mathbb{N})$, $\max(a \cup b) = \max(\max(a), \max(b))$, so $g(a \cup b) = f([0, \max(\max(a), \max(b))]) = f([0, \max(a)]) \cap f([0, \max(b)]) = g(a) \cap g(b)$. □

Definition 3. A filter $\mathcal{F}$ on κ is called *countably incomplete* if there are sets $V_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $\bigcap_{n \in \mathbb{N}} V_n = \varnothing$.

Note that any nonprincipal filter on $\mathbb{N}$ is countably incomplete, so there are lots of countably incomplete good ultrafilters on $\mathbb{N}$. However, for $\kappa > \aleph_0$, the existence of such ultrafilters on κ is a nontrivial result:

Theorem 4 (Kunen, [Kun72]). *Any infinite cardinal κ admits a countably incomplete good ultrafilter.*

Theorem 5 (Keisler, [Kei10, 10.5]). *Let τ be a signature with $|\tau| \leq \kappa$ and let $(\mathbf{M}_\lambda)_{\lambda<\kappa}$ be a sequence of τ-structures. For any countably incomplete good ultrafilter $\mathcal{U}$ on κ, the ultraproduct $\mathbf{M} := \prod_{\lambda \to \mathcal{U}} \mathbf{M}_\lambda$ is κ^+-saturated.*

Proof. Take $A \subseteq M$ with $|A| \leq \kappa$ and let $\{D_i\}_{i \in I}$ be a set of distinct A-definable subsets of $\mathbf{M}$ with the finite intersection property (FIP); our goal is to show that there is $d \in M$ that belongs to all of D_i. Since there are at most $\kappa = \max(\aleph_0, |\tau|, |A|)$-many distinct A-definable sets, we may assume that $I = \kappa$. For every $a \in \mathscr{P}_{\text{fin}}(\kappa)$, we have

$$\bigcap_{\alpha \in a} D_\alpha \neq 0,$$

so Łoś's theorem gives

$$\forall^{\mathcal{U}} \lambda < \kappa, \bigcap_{\alpha \in a} D_\alpha^{(\lambda)} \neq 0.$$

In other words, the sets $U_a := \{\lambda < \kappa : \bigcap_{\alpha \in a} D_\alpha^{(\lambda)} \neq \varnothing\}$ are $\mathcal{U}$-large. Letting $(V_n)_{n \in \mathbb{N}} \subseteq \mathcal{U}$ be a decreasing sequence that witnesses the countable incompleteness of $\mathcal{U}$, we define $f : \mathscr{P}_{\text{fin}}(\kappa) \to \mathcal{U}$ by $f(a) := U_a \cap V_{|a|}$ and note that f is monotone. Now the goodness of $\mathcal{U}$ gives a homomorphism $g : \mathscr{P}_{\text{fin}}(\kappa) \to \mathcal{U}$ with $g(a) \subseteq f(a)$ for each $a \in \mathscr{P}_{\text{fin}}(\kappa)$. Let $U = \bigcup_{a \in \mathscr{P}_{\text{fin}}(\kappa)} g(a)$; clearly $U \in \mathcal{U}$. For each $\lambda \in U$, let $G_\lambda := \{a \in \mathscr{P}_{\text{fin}}(\kappa) : \lambda \in g(a)\}$.

Claim. Let $\lambda \in U$.

(a) G_λ is a directed family with respect to inclusion; in fact, $a, b \in G_\lambda \Rightarrow a \cup b \in G_\lambda$.
(b) G_λ is finite.
(c) There is a $\subseteq$-maximum $a_\lambda \in G_\lambda$, namely, $a_\lambda = \bigcup G_\lambda$.

Proof of Claim. Part (a) follows from the fact that g is a homomorphism.

For (b), note that otherwise, using part (a), we would get a strictly increasing chain $(a_k)_{k\in\mathbb{N}} \subseteq G_\lambda$, so, in particular $\lambda \in V_{|a_k|}$ for each k, and hence $\lambda \in \bigcap_k V_{|a_k|}$. But because $|a_k| \to \infty$ as $k \to \infty$ and the sets $(V_n)_{n\in\mathbb{N}}$ are decreasing, the latter intersection is equal to $\bigcap_n V_n = \varnothing$, a contradiction.

Part (c) follows immediately from (a) and (b). $\dashv$

Finally, we are ready to define $d \in M$ and it is enough to only specifying its values on U. For each $\lambda \in U$, let $d(\lambda)$ be any element of $\bigcap_{\alpha\in a_\lambda} D^\lambda_\alpha$ (we use the AC here, but this could be avoided). Fixing $\alpha < \kappa$, we now check that $d \in D_\alpha$. Indeed, for each $\lambda \in g(\{\alpha\})$, $a_\lambda \ni \alpha$, so $d(\lambda) \in D^\lambda_\alpha$ by definition. Because $g(\{\alpha\})$ is $\mathcal{U}$-large, we get that $\forall^{\mathcal{U}} \lambda < \kappa \; d(\lambda) \in D^\lambda_\alpha$, and hence, $d \in D_\alpha$. □

References

[Kei10] H. J. Keisler, *The ultraproduct construction*, Ultrafilters Accross Mathematics (V. Bergelson et. al., ed.), Contemporary Mathematics, vol. 530, Amer. Math. Soc., 2010, available at http://www.math.wisc.edu/~keisler/ultraproducts-web-final.pdf.

[Kun72] K. Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. **172** (1972), 199–206.