MATH 570: MATHEMATICAL LOGIC

HOMEWORK 9

Due date: Nov 5 (Wed)

1. Determine which of the following statement are true for an arbitrary τ_{a}-sentence θ and prove your answers:

(a) $\mathsf{PA} \vdash \theta \iff \mathbf{N} \vDash \mathbf{Provable}_{\mathsf{PA}}([\theta])$,

(b) $\mathsf{PA} \vdash \theta \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,

(c) $\mathsf{PA} \vdash \theta \implies \mathsf{PA} \vdash \mathbf{Provable}_{\mathsf{PA}}([\theta])$.

2. Let ϕ and θ be τ_a-sentences. Consider the following statements:

(1) $\mathsf{PA} \vdash \phi \implies \mathsf{PA} \vdash \theta$;

(2) $\mathsf{PA} \vdash \phi \to \theta$.

Are they equivalent for all ϕ, θ? If not, which implication holds and which may fail? Prove your answers.

3. Let Determine which of the following τ_{a}-sentences are provable in PA for an arbitrary τ_{a}-sentence θ:

(a) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \theta$.

(b) $\mathbf{Provable}_{\mathsf{PA} \cup \{\neg\theta\}}([\theta]) \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,

(c) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \neg\mathbf{Provable}_{\mathsf{PA}}([\neg\theta])$,

(d) $\mathbf{Provable}_{\mathsf{PA}}(\mathbf{Provable}_{\mathsf{PA}}([\theta])) \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,

(e) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \mathbf{Provable}_{\mathsf{PA}}(\mathbf{Provable}_{\mathsf{PA}}([\theta]))$.

Prove all your answers except for part (e). For the latter, just make a guess.

4. Show that the set of Σ^0_1 relations is closed under finite unions/intersections and taking projections, i.e. under the operations $\vee, \wedge, \exists$.

5. For $A \subseteq \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, we say that f *enumerates* A if the image of f is A, i.e. $f[\mathbb{N}] = A$.

Definition. A set $A \subseteq \mathbb{N}$ is called *recursively enumerable* (r.e. for short) if there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ enumerating A.

Prove the following characterizations of recursive and Σ^0_1 sets:

(a) A set $A \subseteq \mathbb{N}$ is recursive if and only if it is either finite or enumerable by a *strictly increasing* recursive function.

(b) For a nonempty set $A \subseteq \mathbb{N}$, the following are equivalent:

   (1) A is Σ^0_1

   (2) A is either finite or enumerable by an *injective* recursive function

   (3) A is r.e.

6. Let T be a τ-theory, where τ is a finite signature.

(a) Prove that if T is decidable, then it has a recursive completion.[1]

(b) Deduce Church's theorem: Any τ-theory that interprets Q is undecidable.

7. Prove **Craig's lemma**, namely, that any Σ_1^0 theory T (in a finite signature τ) has a recursive axiomatization. Conclude that we CAN replace "recursive" by "Σ_1^0" in Rosser's form of the First Incompleteness theorem.

HINT: For a sentence ϕ, ϕ being in T is "recursively witnessed" by a number $n_\phi \in \mathbb{N}$. Modify ϕ into a logically equivalent sentence that encodes the witness n_ϕ.

REMARK 1: Recall that we couldn't replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.

REMARK 2: One can in fact show that any Σ_1^0 theory has a *primitive recursive* axiomatization.

8. Let $A, B \subseteq \mathbb{N}^k$. Show the following:

(a) (Reduction property for Σ_1^0) If A, B are Σ_1^0, then there are disjoint Σ_1^0 sets $A^*, B^* \subseteq \mathbb{N}^k$ such that $A^* \subseteq A$, $B^* \subseteq B$ and $A^* \cup B^* = A \cup B$.

(b) (Separation property for Π_1^0) If A, B are disjoint Π_1^0 sets, then there is a Δ_1^0 (and hence recursive) set $S \subseteq \mathbb{N}^k$ such that $S \supseteq A$ and $S \cap B = \varnothing$.

[1] Many thanks to Anton Bernshteyn for suggesting this question.