MATH 570: MATHEMATICAL LOGIC

HOMEWORK 8

Due date: Oct 22 (Wed)

1. Let $n, m \in \mathbb{N}$. Show the following:

(a) $\neg n \le m \iff \mathsf{Q} \vdash \neg\Delta(n) \le \Delta(m)$;

HINT: For $\Longleftarrow$ show the contrapositive.

(b) $\mathsf{Q} \vdash x \le \Delta(n) \vee \Delta(n+1) \le x$.

HINT: Prove by induction on n.

2. (a) Show that the representability of recursive functions in Q implies that recursive functions/relations are arithmetical.

(b) Give a direct proof that recursive functions/relations are arithmetical (without using their representability in Q).

3. Show that Gödel's Incompleteness theorem (the original form) is equivalent to the statement that $\mathsf{Th}(\mathbf{N})$ is not recursive.

4. (a) Show that we can replace "recursive" by "arithmetical" in the statement of Gödel's Incompleteness theorem (the original form), i.e. prove that if $T \subseteq \mathsf{Th}(\mathbf{N})$ is arithmetical, then it is incomplete.

(b) Show that there exists an arithmetical completion of PA, i.e. there is a complete τ_{a}-theory $T \supseteq PA$ such that $\ulcorner T \urcorner = \{\ulcorner \phi \urcorner : \phi \in T\}$ is an arithmetical subset of $\mathbb{N}$. Conclude that we CANNOT replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.