MATH 570: MATHEMATICAL LOGIC

HOMEWORK 7

Due date: Oct 15 (Wed)

1. Choose any two of the following facts about the Ackermann function and prove them:

(a) $A(n, x + y) \geq A(n, x) + y$;

(b) $n \geq 1 \implies A(n + 1, y) > A(n, y) + y$;

(c) $A(n + 1, y) \geq A(n, y + 1)$;

(d) $2A(n, y) < A(n + 2, y)$;

(e) $x < y \implies A(n, x + y) \leq A(n + 2, y)$.

2. Show that the graph of the Ackermann function is primitive recursive.

Hint: Use a similar argument to the proof that the Ackermann function is recursive, but bound your search using the value of the function.

Note: This implies that the Ackermann function is recursive.

3. (a) Show that the Ackermann function grows faster than any primitive recursive function; more precisely, prove that for any primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists $n_f \in \mathbb{N}$ such that $f(\vec{x}) \leq A(n_f, \|\vec{x}\|_1)$ for all $\vec{x} \in \mathbb{N}^k$, where $\|\vec{x}\|_1 = x_1 + \dots + x_n$.

Hint: Use Problem 1.

(b) Conclude that the Ackermann function is not primitive recursive.

4. Prove the following proposition using the outline below.

Proposition. *There exists a recursive function $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that for every n, $\phi_n := \phi(n, \cdot)$ is primitive recursive and for every $k \in \mathbb{N}$ and every primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there is n such that $\forall \vec{a} \in \mathbb{N}^k$,*

$$f(\vec{a}) = \phi_n(\langle \vec{a} \rangle).$$

In the latter case, we say that ϕ_n corresponds to f.

In this proposition and below, the tuple coding function $\langle\rangle$ is assumed to satisfy

$$\langle a_0, \dots, a_{k-1} \rangle \geq \max\{k, a_0, \dots, a_{k-1}\}.$$

Outline: Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be a function with the following property: for every $n \in \mathbb{N}$,

- if $n = \langle 1, a, m \rangle$ then ϕ_n corresponds to
  - the function $S : \mathbb{N} \to \mathbb{N}$ if $m = 0$ and $a = 1$;
  - the function $P^a_{(i-1)/2} : \mathbb{N}^a \to \mathbb{N}$ if $m = \langle k, i \rangle$, $a = k$ and i is odd;
  - the function $C^a_{i/2} : \mathbb{N}^a \to \mathbb{N}$ if $m = \langle k, i \rangle$, $a = k$ and i is even.
- if $n = \langle 2, a, m \rangle$, where
  - $m = \langle n_0, \dots, n_k \rangle$ for some $k \geq 1$,
  - $(n_0)_1 = k$,
  - $(n_i)_1 = a$ for $i = 1, \dots, k$,

then letting $g : \mathbb{N}^k \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h_i : \mathbb{N}^a \to \mathbb{N}$ the functions corresponding to ϕ_{n_i}, (for $i = 1, ..., k$), ϕ_n corresponds to the function obtained by *composition* from g and $h_0, ..., h_{k-1}$.

- if $n = \langle 3, a, m \rangle$, where
  - $a \geq 1$
  - $m = \langle n_0, n_1 \rangle$
  - $(n_0)_1 = a - 1$,
  - $(n_1)_1 = a + 1$,

  then letting $g : \mathbb{N}^{a-1} \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h : \mathbb{N}^{a+1} \to \mathbb{N}$ the function corresponding to ϕ_{n_1}, ϕ_n corresponds to the function obtained by *primitive recursion* from g and h.

Note that to calculate the value $\phi(n, l)$, one needs to know $\phi(n', l')$ for only finitely many (n', l') with either $n' < n$ or $l' < l$. Use this and an idea similar to Dedekind's analysis of recursion to show that ϕ is recursive (i.e. one can define a recursive ϕ satisfying the property above).

5. Prove the following:

(a) $\mathsf{Q} \nvdash (x + y) + z = x + (y + z)$;

(b) $\mathsf{Q} \nvdash x + y = y + x$;

(c) $\mathsf{Q} \nvdash \forall x(0 + x = x)$.