MATH 570: MATHEMATICAL LOGIC

HOMEWORK 6

Due date: Oct 8 (Wed)

1. Explain why Tarski's theorem (on $\mathsf{Th}(\mathbf{N})$ not being arithmetical) is equivalent to the fixed point lemma.

2. Sketch the proof of Gödel's Incompleteness Theorem. I will ask you to present it on the board.

3. Prove Lemma 5.8(e) as well as Lemma 5.14(a,c,e).

4. Put $\mathbb{N}^{<\mathbb{N}} := \bigcup_{l\in\mathbb{N}} \mathbb{N}^l$ and let $\langle\rangle : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a primitive recursive coding of tuples; for example,

$$\langle n_1, ..., n_l \rangle = p_1^{n_1+1} ... p_l^{n_l+1},$$

where $l \in \mathbb{N}$ and p_i denotes the i^{th} prime number. For $f : \mathbb{N}^{k+1} \to \mathbb{N}$, define

$$\bar{f}(\vec{a}, n) = \langle f(\vec{a}, 0), f(\vec{a}, 1), ..., f(\vec{a}, n-1) \rangle.$$

Given $h : \mathbb{N}^{k+2} \to \mathbb{N}$, let $f : \mathbb{N}^2 \to \mathbb{N}$ be defined by the identity

$$f(\vec{a}, n) = h(\vec{a}, \bar{f}(\vec{a}, n)).$$

Show that if h is primitive recursive, then so is f.

5. Let $g : \mathbb{N} \to \mathbb{N}$, $h : \mathbb{N}^3 \to \mathbb{N}$, $\tau : \mathbb{N}^2 \to \mathbb{N}$. We say that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by *nested recursion* from g, h, τ if for each $x, y \in \mathbb{N}$,

$$\begin{cases} f(0, y) &= g(y) \\ f(x+1, y) &= h(x, y, f(x, \tau(x, y))) \end{cases}.$$

Show that if g, h, τ are primitive recursive, then so is f.