# MATH 570: MATHEMATICAL LOGIC

HOMEWORK 5

Due date: Oct 1 (Wed)

**1.** Follow the steps below to show that the class $\mathcal{D}$ of disconnected graphs is not axiomatizable (this would've been quite a tricky problem without the steps). Assume for contradiction that there is an axiomatization $T$ of $\mathcal{D}$ in some signature $\tau$. Let $\tau' := \tau \cup \{u, v\}$, where $u, v$ are new constants and put

$$S := \{\chi_n(u, v) : n \in \mathbb{N}\},$$

where the formula $\chi_n(x, y)$ says that there is no path of length $\leq n$ between $x$ and $y$ (here $x, y$ are variables).

(i) Show that for every $\phi \in T$ there is $n \in \mathbb{N}$ such that $\chi_n(u, v) \vDash \phi$.

(ii) Conclude that $\exists x \exists y \chi_n(x, y) \vDash \phi$, for $\phi$ and $n$ as in (i).

(iii) Put $S' := \{\exists x \exists y \chi_n(x, y) : n \in \mathbb{N}\}$ and conclude that $S' \vDash T$, i.e. for every $\phi$, $S' \vDash \phi$. Explain why this is a contradiction.

(iv) In a nutshell, what was the idea of the proof and what was the crucial step?

**2.** The following is a well known theorem of additive combinatorics:

**Theorem** (van der Waerden)**.** *Suppose $\mathbb{Z}$ is finitely colored. Then one of the color classes contains arbitrarily long arithmetic progressions.*

Use this theorem and the Compactness theorem to derive the following finitary version:

**Theorem.** *Given any positive integers $m$ and $k$, there exists $N \in \mathbb{N}$ such that whenever $\{0, 1, ..., N-1\}$ is colored with $m$ colors, one of the color classes contains an arithmetic progression of length $k$.*

**3.** Let DLO denote the theory of dense linear orderings without endpoints as defined in Example 4.2(b) of the lecture notes.

(a) Show that DLO is $\aleph_0$-categorical, and hence $(\mathbb{Q}, <)$ is the only (up to isomorphism) countable dense linear ordering without endpoints.

Hint: Enumerate the two models and recursively construct a sequence of finite partial isomorphisms by going back and forth between the models.

(b) Let $\tau_n := (<, c_1, ..., c_n)$, where $c_i$ are constant symbols. Show that the theory

$$\mathrm{DLO}_n = \mathrm{DLO} \cup \{c_i < c_{i+1} : i < n\}$$

is $\aleph_0$-categorical. Conclude that $\mathrm{DLO}_n$ is complete.

(c) Let $\tau_\infty = (<, \{c_i : i \in \mathbb{N}\})$, where $c_i$ are constant symbols. Show that the theory

$$\mathrm{DLO}_\infty = \mathrm{DLO} \cup \{c_i < c_{i+1} : i \in \mathbb{N}\}$$

is complete.

(d) Yet, show that $\mathrm{DLO}_\infty$ has exactly three countable nonisomorphic models and hence is not $\aleph_0$-categorical.

**4.** Let $K$ be a field and let $\overline{K}$ be an algebraic closure of $K$. A nonconstant polynomial $f \in K[X_1, ..., X_n]$ is called *irreducible over* $K$ if whenever $f = gh$ for some $g, h \in K[X_1, ..., X_n]$, either $\deg(g) = 0$ or $\deg(h) = 0$. Furthermore, $f$ is called *absolutely irreducible* if it is irreducible over $\overline{K}$.

For example, the polynomial $X^2+1 \in \mathbb{R}[X]$ is irreducible over $\mathbb{R}$, but it is not absolutely irreducible since $X^2 + 1 = (X + i)(X - i)$ in $\mathbb{C}[X]$. On the other hand, $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and prove the following:

**Theorem** (Noether-Ostrowski Irreducibility Theorem)**.** *For $f \in \mathbb{Z}[X_1, ..., X_n]$ and prime $p$, let $f_p$ denote the polynomial in $\mathbb{F}_p[X_1, ..., X_n]$ obtained by applying the canonical map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to the coefficients of $f$ (i.e. moding out the coefficients by $p$). For all $f \in \mathbb{Z}[X_1, ..., X_n]$, $f$ is absolutely irreducible (as an element of $\mathbb{Q}[X_1, ..., X_n]$) if and only if for sufficiently large primes $p$, $f_p$ is absolutely irreducible (as an element of $\mathbb{F}_p[X_1, ..., X_n]$).*

Hint: Your proof should be shorter than the statement of the problem.

Remark: The original algebraic proof of this theorem is quite involved!