MATH 570: MATHEMATICAL LOGIC

HOMEWORK 4

Due date: Sep 24 (Wed)

1. Show that if a theory has arbitrarily large finite models, then it has an infinite model.

2. (Weak Lefschetz Principle) Let ϕ be a τ_{ring}-sentence. Show that if $\mathsf{ACF}_0 \vDash \phi$, then for large enough primes p, $\mathsf{ACF}_p \vDash \phi$.

3. Prove that a graph is 3-colorable if and only if so is every finite subgraph of it.

HINT: To be able to express the 3-colorability property, add to the signature of graphs a unary relation symbol for each color. Also, add a name (constant symbol) for every vertex in the graph.

4. Show that the class of connected graphs is not axiomatizable.

5. Let $\mathbf{M} \vDash \mathsf{PA}$.

(a) Let $\mathbf{N} := (\mathbb{N}, 0^{\mathbf{N}}, S^{\mathbf{N}}, +^{\mathbf{N}}, \cdot^{\mathbf{N}})$ denote the standard model of PA, i.e. the usual natural numbers. Show that there is a unique homomorphism $f : \mathbf{N} \to \mathbf{M}$ and this f is one-to-one, and hence (why?) is a τ_{a}-embedding.

(b) In the notation of (a), define the standard part of $\mathbf{M}$ by

$$\overline{\mathbb{N}} = f[\mathbb{N}].$$

Show that if $\mathbf{M}$ is nonstandard then $\overline{\mathbb{N}}$ is not definable in $\mathbf{M}$.

(c) (Overspill) Assume $\mathbf{M}$ is nonstandard, and let $\phi(x, \vec{y})$ be a τ_{a}-formula, where $\vec{y}$ is a k-vector and $\vec{a} \in M^k$. Show that if $\mathbf{M} \vDash \phi(n, \vec{a})$ for infinitely many $n \in \overline{\mathbb{N}}$, then there is $w \in M \setminus \overline{\mathbb{N}}$ such that $\mathbf{M} \vDash \phi(w, \vec{a})$. In other words, if a statement is true about infinitely many natural numbers, then it is true about an infinite number.

6. Give an example of structures $\mathbf{A} \prec \mathbf{B}$ that demonstrate the failure of the converse of Problem 1 of HW3.

7. This problem illustrates the (crazy) structure of the nonstandard models of PA. Let $\mathbf{M}$ be a nonstandard model of PA and let $\overline{\mathbb{N}}$ be its standard part. By replacing it with $\mathbb{N}$, we can assume without loss of generality that $\overline{\mathbb{N}} = \mathbb{N}$.

(a) For $a, b \in M$, let $|a - b|$ denote the unique $d \in M$ such that $a + d = b$ or $b + d = a$ (why does such exist?). For all $a, b \in M$, define

$$a \sim b :\Leftrightarrow |a - b| \in \mathbb{N}.$$

Show that $\sim$ is an equivalence relation on M and that it is NOT definable in $\mathbf{M}$.

(b) Put $Q = M/\sim$, so $Q = \{[a] : a \in M\}$, where $[a]$ denotes the equivalence class of a. Define the relation $\leq_Q$ on Q as follows: for all $[a], [b] \in Q$,

$$[a] <_Q [b] \iff \exists c \in M \setminus \mathbb{N} \text{ such that } a + c = b$$

Show that $<_Q$ is well-defined (does not depend on the representatives a, b) and is a linear ordering on Q.

(c) Show that the ordering $(Q, \leq_Q)$ has a least element but no greatest element; moreover, show that it is a dense (in itself), i.e. for all $u, v \in Q$,

$$u <_Q v \implies \exists w (u <_Q w <_Q v),$$

where $u <_Q v$ stands for $u \leq_Q v \wedge u \neq v$.