MATH 570: MATHEMATICAL LOGIC

HOMEWORK 3

Due date: Sep 17 (Wed)

1. Let $\mathbf{A} \subseteq \mathbf{B}$ and assume that for any finite $S \subseteq A$ and $b \in B$, there exists an automorphism f of $\mathbf{B}$ that fixes S pointwise (i.e. $f(a) = a$ for all $a \in S$) and $f(b) \in A$. Show that $\mathbf{A} \prec \mathbf{B}$.

2. Show that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$, but $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$.

Hint: Use the previous problem.

3. Conclude from the Löwenheim-Skolem theorem that any satisfiable theory T has a model of cardinality at most $\max\{|\tau|, \aleph_0\}$. In particular, if ZFC is satisfiable, then it has a countable model (although that model $\mathbf{M}$ would believe it contains sets of uncountable cardinality, e.g. $\mathbb{R}^{\mathbf{M}}$). Explain why this DOES NOT imply that ZFC is not satisfiable.

4. Prove the Constant Substitution lemma.

5. Show the following:

(a) (Associativity of +) $\mathsf{PA} \vdash \forall x \forall y \forall z[(x + y) + z = x + (y + z)]$,

(b) $\mathsf{PA} \vdash \forall x(0 + x = x)$,

(c) (Commutativity of +) $\mathsf{PA} \vdash \forall x \forall y(x + y = y + x)$.

6. Show that a τ-theory T is semantically complete if and only if for any $\mathbf{A}, \mathbf{B} \models T$, $\mathbf{A} \equiv \mathbf{B}$.

7. For a fixed signature τ, let $\mathcal{T}$ denote the set of all consistent complete theories and we equip this set with the topology generated by the sets $\langle \phi \rangle := \{T \in \mathcal{T} : T \vdash \phi\}$, as described in the lecture notes. Show that for a τ-theory T (this T is not necessarily consistent or complete), the following are equivalent (i.e. can be deduced one from another):
(1) $T \vdash \phi$ implies that there is finite $T_0 \subseteq T$ with $T_0 \vdash \phi$.
(2) T being inconsistent implies that some finite $T_0 \subseteq T$ is inconsistent.
(3) $\mathcal{T}$ is compact.

Hint: First prove (1) $\Leftrightarrow$ (2). Then prove (2) $\Leftrightarrow$ (3) using the easy fact that a topological space is compact if and only if every family of closed sets[1] with the finite intersection property[2] has a nonempty intersection. It helps to associate T with the family $\{\langle \phi \rangle : \phi \in T\}$ of clopen sets. The fact that any consistent theory has a consistent completion should also be useful.

8. Let $\mathcal{T}$ be as in the previous problem. For every τ-sentence ϕ, put

$$\langle \phi \rangle = \{T \in \mathcal{T} : T \vdash \phi\}.$$

(a) Prove that for $A \subseteq \mathcal{T}$, if

$$A = \bigcap_{i \in I} \langle \phi_i \rangle = \bigcup_{j \in J} \langle \psi_j \rangle,$$

[1]Here, one can always assume (why?) that these are basic closed sets, i.e. complements of sets in a basis.
[2]A family of sets $\mathcal{F}$ is said to have the *finite intersection property* if for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$, $\bigcap_{A \in \mathcal{F}_0} A \neq \varnothing$.

then there are finite $I_0 \subseteq I, J_0 \subseteq J$ such that

$$A = \bigcap_{i \in I_0} \langle \phi_i \rangle = \bigcup_{j \in J_0} \langle \psi_j \rangle.$$

(b) Conclude from (a) that the only clopen (closed and open) sets in $\mathcal{T}$ are of the form $\langle \phi \rangle$, where ϕ is a τ-sentence.