MATH 570: MATHEMATICAL LOGIC

HOMEWORK 2

Due on Sep 10 (Wed)

1. Let ϕ be a τ-sentence. The *finite spectrum* of ϕ is the set

$$\{n \in \mathbb{N}^+ : \text{there is } \mathbf{M} \vDash \phi \text{ with } |M| = n\},$$

where $\mathbb{N}^+$ is the set of positive integers.

(a) Let $\tau = (E)$, where E is a binary relation symbol, and let ϕ be the sentence that asserts that E is an equivalence relation each class of which has exactly 2 elements. Show that the finite spectrum of ϕ is all positive even numbers.

(b) For each of the following subsets of $\mathbb{N}^+$, show that it is the finite spectrum of some sentence ϕ in some signature τ:

(i) $\{2^n 3^m : n, m \in \mathbb{N}^+\}$,

Hint: Groups...

(ii) $\{n \in \mathbb{N}^+ : n \text{ is composite}\}$,

Hint: Subgroups and cosets... (Although, there are other ways to do this.)

(iii) $\{p^n : p \text{ is prime and } n \in \mathbb{N}^+\}$,

Hint: Fields...

(iv) $\{p : p \text{ is prime}\}$.

Hint: Fields with an ordering...

(v) $\{n^2 : n \in \mathbb{N}^+\}$,

Hint: One way to do this is using the signature (S, f, g), where S is a unary relation symbol (hence will be interpreted as a subset of the universe), and f, g are unary function symbols.

2. Determine whether the following are 0-definable:

(a) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.

Hint: You need a nontrivial fact from elementary number theory.

(b) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.
(c) The set of non-negative numbers in $(\mathbb{Q}, +)$.
(d) The set of positive numbers in $(\mathbb{R}, <)$.
(e) The function $\max(x, y)$ in $(\mathbb{R}, <)$.
(f) The function $\text{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.
(g) 2 in $(\mathbb{R}, +, \cdot)$.
(h) The relation $d(x, y) = 2$ in an undirected graph (with no loops) (Γ, E), where $d(x, y)$ denotes the edge distance function.

Hint: Wondering how to prove the negative answers? Use the fact that isomorphisms (in particular, automorphisms) preserve satisfiability of formulas (see Proposition 2.21 in the notes).

3. (a) Using appropriate signature, define a theory for vector spaces over $\mathbb{Q}$.

(b) Let $\mathcal{M}$ be the class of metric spaces with metric bounded by 1. In the signature $(d_r : r \in [0,1] \cap \mathbb{Q})$, define a theory T for $\mathcal{M}$ in the following sense:
- for every model $\mathbf{M}$ of T, the function $\rho_{\mathbf{M}} : M^2 \to [0,1]$ given by $(x,y) \mapsto \inf_{r \in [0,1] \cap \mathbb{Q}} d_r(x,y)$ is well-defined and is a metric on M, i.e. $(M, \rho_{\mathbf{M}}) \in \mathcal{M}$;
- conversely, for every metric space $(X, \rho) \in \mathcal{M}$, there is a model $\mathbf{M}$ of T such that the universe of $\mathbf{M}$ is equal to X and $\rho_{\mathbf{M}} = \rho$, where $\rho_{\mathbf{M}}$ is defined as above.

4. Find a sentence that is true in $(\mathbb{N}, <)$ and false in $(\mathbb{Z}, <)$.

5. A formula is called *universal* (*existential*) if it of the form $\forall x \psi$ ($\exists x \psi$), where ψ is quantifier free. Let $\mathbf{A} \subseteq \mathbf{B}$ be τ-structures, $\phi(\vec{v})$ be a τ-formula and $\vec{a} \in A^n$. Show that

(a) if ϕ is quantifier free, then $\mathbf{A} \vDash \phi(\vec{a}) \iff \mathbf{B} \vDash \phi(\vec{a})$;

(b) if ϕ is universal, then $\mathbf{B} \vDash \phi(\vec{a}) \implies \mathbf{A} \vDash \phi(\vec{a})$;

(c) if ϕ is existential, then $\mathbf{A} \vDash \phi(\vec{a}) \implies \mathbf{B} \vDash \phi(\vec{a})$.

6. Let $\vec{v_1}, ..., \vec{v_n} \in \mathbb{Q}^m$. Show that $\{\vec{v_1}, ..., \vec{v_n}\}$ is linearly independent over $\mathbb{Q}$ if and only if it is linearly independent over $\mathbb{R}$.

Hint: Show that linear independence can be expressed by both universal and existential formulas.

7*. (Extra problem, not mandatory) Show that the relation

$$P(x,y) \iff x \text{ and } y \text{ are connected}$$

is not 0-definable in the graph (Γ, E) that consists of two bi-infinite paths, i.e. two disjoint copies of $\mathbb{Z}$.

Hint: The idea is to show that any formula $\phi(x_1, ..., x_k)$ of length[1] n does not distinguish between different tuples $(u_1, ..., u_k) \in \Gamma^k$ of vertices that are 2^n-spread out, i.e. the edge-distance $d_e(u_i, u_j)$ between u_i and u_j is at least 2^n or infinite (i.e. no path between a and b), for all $i < j$. Try proving this by induction on the length of ϕ and you'll quickly realize that for the step with $\exists$ to go through you need to be proving a stronger statement. To come up with it, one has to define what it means for two tuples of vertices to have *the same* 2^n*-type*. The latter is defined using the following notion: for a finite set $F \subseteq \Gamma$, vertices $u, v \in F$ are said to be 2^n-equivalent over F if there is a sequence $v_1, ..., v_l \subseteq F$ with $u = v_1$ and $v = v_l$ such that $d_e(v_i, v_{i+1}) < 2^n$ for all $i < l$. I'll be happy to discuss this in my office hour.

[1] Here we view formulas as sequences of symbols and by *length of a formula* we mean the length of the sequence.